POLITEKNIK MALAYSIA

KEMENTERIAN PENGAJIAN TINGGI

KOLEJ KOMUNITI MALAYSIA

SIGNALS AND SYSTEMS

FARIZA ZAHARI
NORAZLINA JAAFAR

ELECTRICAL ENGINEERING DEPARTMENT

SIGNALS AND SYSTEMS
Laplace Transform
Z Transform
Fourier Analysis of Continuous Time
Discrete Time Signals and Systems

Authors:
Fariza Binti Zahari
Norazlina Binti Jaafar

eISBN No: 978-967-2044-82-6

First Published in 2021 by:

UNIT PENERBITAN

Politeknik Sultan Salahuddin Abdul Aziz Shah
Persiaran Usahawan,
Seksyen U1,
40150 Shah Alam
Selangor

Telephone No. : 03 5163 4000
Fax No. : 03 5569 1903

AUTHOR BIODATA

Name	**FARIZA BINTI ZAHARI**
Designation	Lecturer
Department	Department of Electrical Engineering
E-mail Address	fariza@psa.edu.my
Academic Qualification	Bachelor in Electrical Engineering (Medical Electronic), Kolej Universiti Teknologi Tun Hussein Onn (KUiTTHO) Master in Technical & Vocational Education, Kolej Universiti Teknologi Tun Hussein Onn (KUiTTHO)

Name	Norazlina binti Jaafar
Designation	Lecturer
Department	Department of Electrical Engineering
E-mail Address	jnorazlina@psa.edu.my
Academic Qualification	Master Sc. in Communication and Networking Engineering, MARA University of Technology (UiTM) Master in Technical & Vocational Education, Kolej Universiti Teknologi Tun Hussein Onn (KUiTTHO) Bachelor in Electrical Engineering, Kolej Universiti Teknologi Tun Hussein Onn (KUiTTHO) Diploma in Electrical Engineering (Communication), POLIMAS Certificate in Electrical Engineering (Communication), Politeknik Port Dickson

PREFACE

The purpose of this e-book is to provide concepts and theories of signals and systems for the topics of Laplace Transform, Z-transform and Fourier Analysis for continuous and discrete signals that required in almost all fields of electrical engineering and in many other engineering and scientific disciplines as well.

This e-book is intended to be used as an additional learning material developed in electronic form to facilitate access to users. It is also in line to apply the latest technology in line with the changing times.

The concepts in this e-book are topical in each chapter and followed by sample questions and solutions for all the questions given. At the end of each chapter will be given a set of tutorial questions to test the user's level of understanding.

Chapter 1 exploring transformation techniques for LTI system analysis. The Laplace transformation and its application to continuous time LTI system. In addition, this chapter also touches on the Z-transform and its application to discrete time LTI system. Chapter 2 discusses Fourier Analysis, Fourier Transform and Frequency Response of signals and systems, example questions and solutions for continuous time signals and discrete time signals.

It is hoped that students will be able to complete the tutorial questions given at the end of each chapter by referring to the examples of solution questions that have been given.

ACKNOWLEDGEMENT

We would like to express our deepest appreciation to all those who have helped us throughout the writing of this e-book.

First of all, thanks to the co-author Dr. Marlina binti Ramli and Pn. Emy Satira Azrin binti Mohamad Hakke where they have also completed part of the topic of this e-book. Not to be forgotten are the friends who gave encouragement and shared ideas and opinions in producing this Signal and System e-book.

Secondly, thanks to the Future Instructional Technology and Academic Centre (FITAC) which has provided many ideas and insights to produce an e-book that is useful to all parties, especially to students.

We are also very indebted to the management of Sultan Salahuddin Abdul Aziz Shah Polytechnic for giving us the opportunity to produce this e-book and always provide support and encouragement for the production of this e-book Signal and System.

Next, unwavering appreciation to the Director of Sultan Salahuddin Abdul Aziz Shah Polytechnic who has created a culture that actively promotes the learning, research, and intellectual development of lecturers on current technological needs.

Finally, it is hoped that the e-book Signal and System can fully benefit students in particular and others in general.

CONTENTS

TOPIC 1

Introduction 4
Laplace Transform for Common Signal 7
Region of Convergence 9
Properties of Laplace Transform 17
Inverse Laplace Transform 24
System Function 29

TOPIC 2

Introduction 36
Z Transform 37
Z Transform Pairs 39
Properties of ROC 43
Inverse Z Transform 47

TOPIC 3

Introduction 53
Periodic Signal 54
Complex Exponential Fourier Series 55
Trigonometric FS 57
Discrete Fourier Series 58
Fourier Transform 62
Properties of Fourier Transform 67
Frequency Response 70

TOPIC 1

Laplace Transform

INTRODUCTION

The Laplace transform is a generalisation of the Fourier transforms for continuous signals that includes sinusoids with exponentially increasing amplitudes in the set of basis signals. The Laplace transform converts a differential equation (real variable t: time) into easier to manipulate and solve algebraic equations (function of complex variable s: frequency).

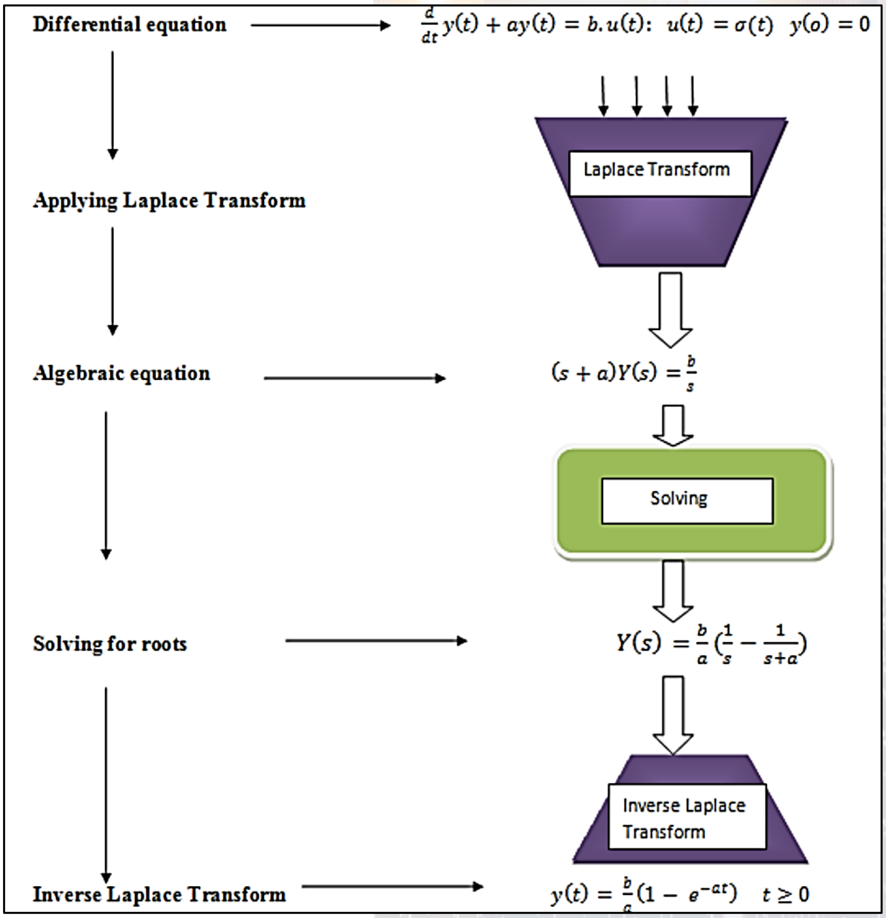


Figure 3.1
Laplace Transform in Real World Problem

The Laplace Transform is a frequency domain to time domain mapping.

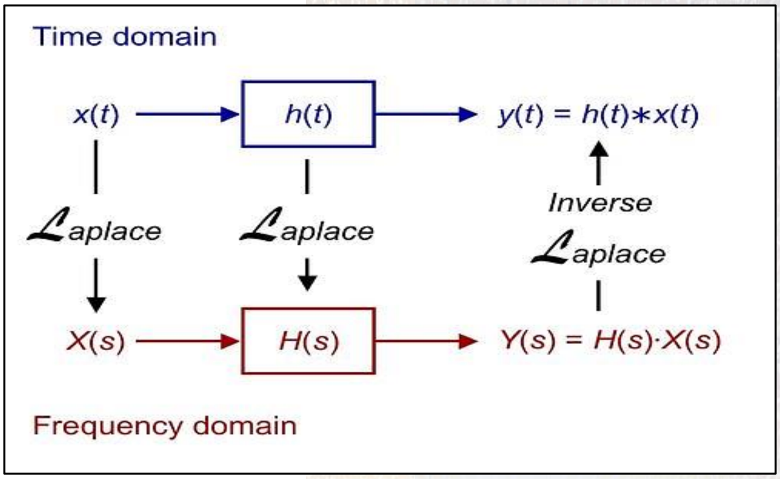
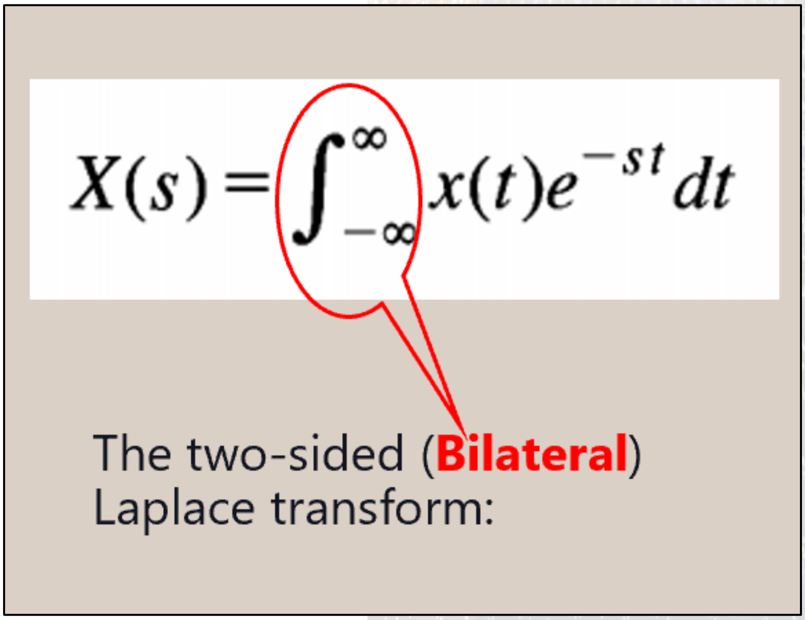


Figure 3.2
The Relationship Between the Time and Frequency Domains

DEFINITION

A function's Laplace Transform, y(t), is defined as:



$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

Figure 3.3
Bilateral Laplace Transform

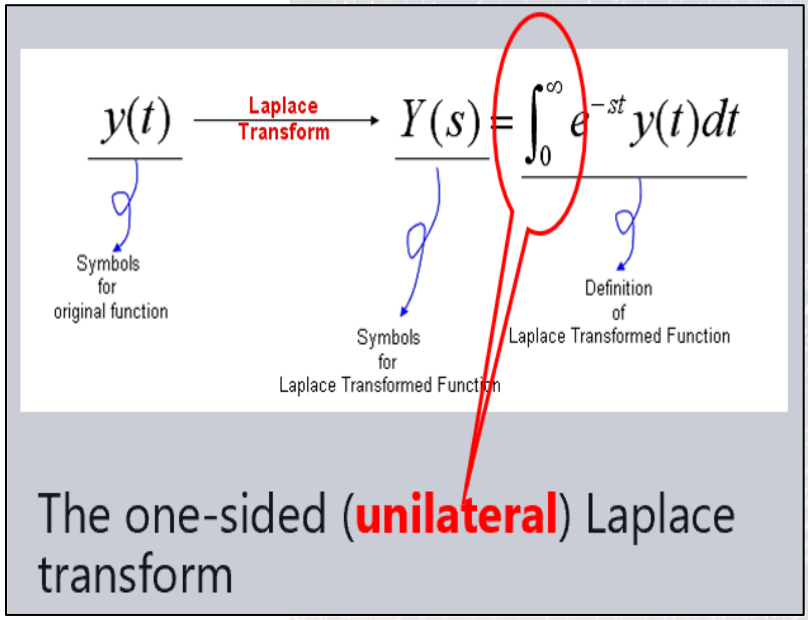


Figure 3.4
Unilateral Laplace Transform

Example 3.1

Find the actual exponential signal's Laplace transform, $x(t) = e^{-at}u(t)$

Solution

Utilizing the Laplace change definition, we get

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^\infty$$

Thus, we obtain

$$= \frac{1}{s+a} \qquad Re(s) > -a$$

Example 3.2

Calculate the Laplace transform $x(t) = -e^{-at}u(-t)$

Solution

Utilizing the Laplace change definition, we get

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}dt = -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

We obtain:

$$= -\left[-\frac{1}{s+a}e^{-(s+a)t}\right]_{-\infty}^{0} = \frac{1}{s+a}$$

LAPLACE TRANSFORM PAIRS FOR COMMON SIGNALS

Table 3-1 lists the Laplace transforms of several typical signals. We can refer to such a table and read out the desired transform instead of having to re-evaluate the transform of a given signal.

Table 3-1 Laplace Transform Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$t^k u(t)$	$\frac{k!}{s^{k+1}}$	$\text{Re}(s) > 0$
$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -\text{Re}(a)$
$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -\text{Re}(a)$
$te^{-at} u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -\text{Re}(a)$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) < -\text{Re}(a)$
$\cos \omega_0 t u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\text{Re}(s) > 0$
$\sin \omega_0 t u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}(s) > 0$
$e^{-at} \cos \omega_0 t u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$
$e^{-at} \sin \omega_0 t u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}(s) > -\text{Re}(a)$

Noted. Adapted from Signal and System, by Hwei, 2010, p. 101.

THE REGION OF CONVERGENCE (ROC)

The range of values of the complex variables s for which the Laplace transform converges is referred to as **region of convergence (ROC)**. Convergence Region The ROC is significant because it determines the area in which the Laplace Transform can be found. In the complex plane, the ROC is usually represented as a separating line/curve. The contour between the regions of convergence and divergence in a continuous summation, such as the Laplace transform or the Fourier transform, is a straight line. The lines are parallel to the imaginary axis for the Laplace transform for which is not zero.

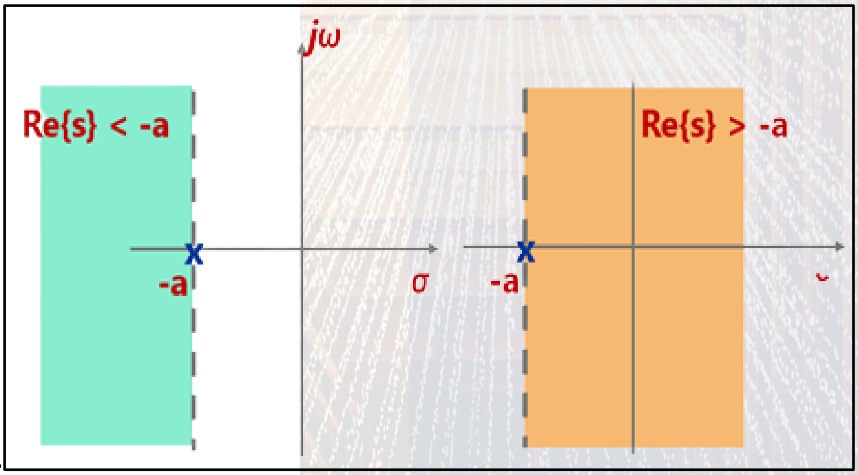


Figure 3.5
The ROC for the Laplace transform is indicated by the shaded regions.

POLES AND ZEROS OF X(S)

X(s) is usually a rational function in s; that is,

$$X(s) = \frac{N(s)}{D(s)} \begin{matrix} \rightarrow \text{zeros} \\ \rightarrow \text{poles} \end{matrix}$$

where D is the denominator polynomial and N is the numerator polynomial.

The pole-zero plot of X(s) is a graphic display of X(s) through its poles and zeros in the s-plane (s). Conventionally, each pole is represented by a "x," while each zero is represented by a "o." This is seen in Figure 3-3 for X(s) provided.

X(s) has one zero at s = -2 and two poles at s = -1 ands = -3 with scale factor 2.

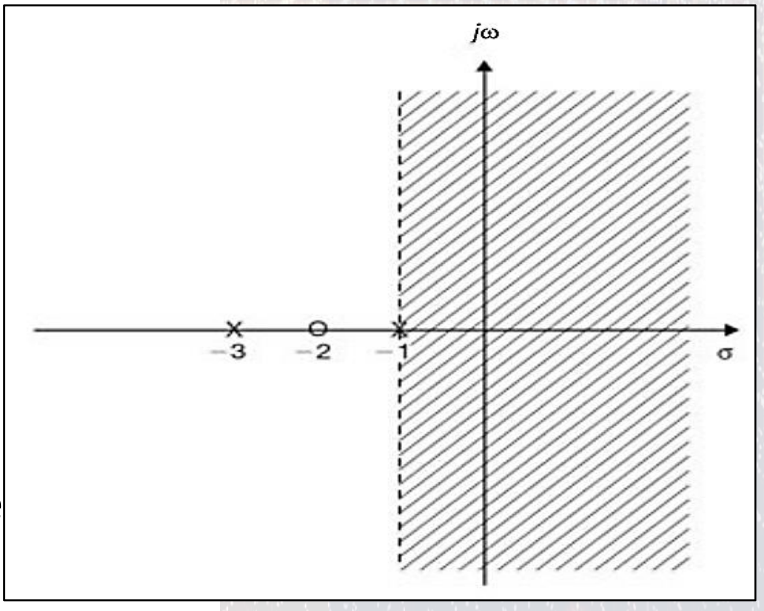


Figure 3.6

An illustration of in the s-plane

X(s) = (2s + 4)/(s 2 + 4s + 3).

Example 3.3

Determine the Laplace transform X(s) and plot the pole-zero with the ROC for $x(t) = e^{-2t}u(t) + e^{-3t}u(t)$

Solution From table 3-1

$$e^{-2t}u(t) \mapsto \frac{1}{s+2} \qquad Re(s) > -2$$

$$\mathrm{e}^{-3\mathrm{t}}\mathrm{u}(\mathrm{t}) \rightarrow \frac{1}{\mathrm{s}+3} \qquad \mathrm{Re}(\mathrm{s}) > -3$$

$$x(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{s+3+s+2}{(s+2)(s+3)}$$

$$= \frac{2s+5}{(s+2xs+3)} = \frac{2\left(s+\frac{5}{2}\right)}{(s+2)(s+3)} \quad , \mathrm{Re}(s) > -2$$

X(s) has two poles at s = - 2 and s = - 3 and one zero at s = - $\frac{5}{2}$ that the ROC is Re(s) > - 2, as plotted in Figure 3.6(a)

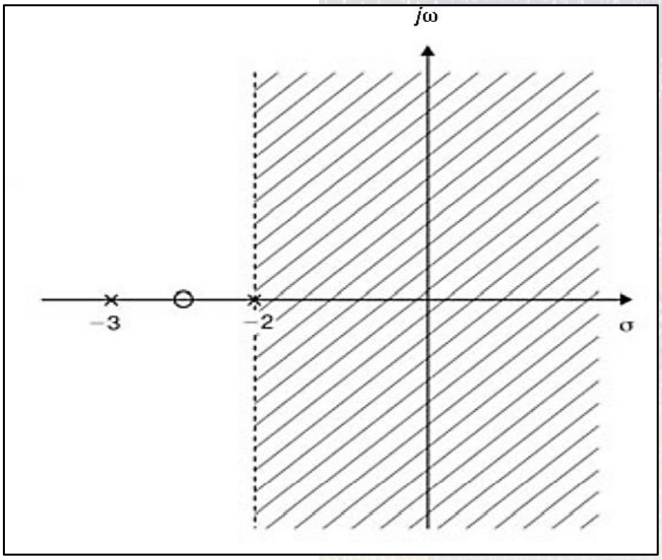


Figure 3.6 (a)

Example 3.4

Discover the worth the Laplace transform X(s) and plot the pole-zero with the ROC for $x(t) = e^{-3t}u(t) + e^{2t}u(-t)$

Solution From table 3-1

$$\mathrm{e}^{-3t}\mathrm{u}(t) \to \frac{1}{\mathrm{s}+3} \qquad \mathrm{Re}(\mathrm{s}) > -3$$

$$e^{2t}u(-t) \mapsto -\frac{1}{s-2} \qquad Re(s) < -2$$

$$X(s) = \frac{1}{s+3} - \frac{1}{s-2} = \frac{s-2-s-3}{(s+3)(s-2)}$$

$$= -\frac{5}{(s+3)(s-2)}$$

X(s) has two poles at s = 2 ands = - 3 and no zeros, that the ROC is - 3 < Re(s) < 2, as plotted in Figure 3.6 (b)

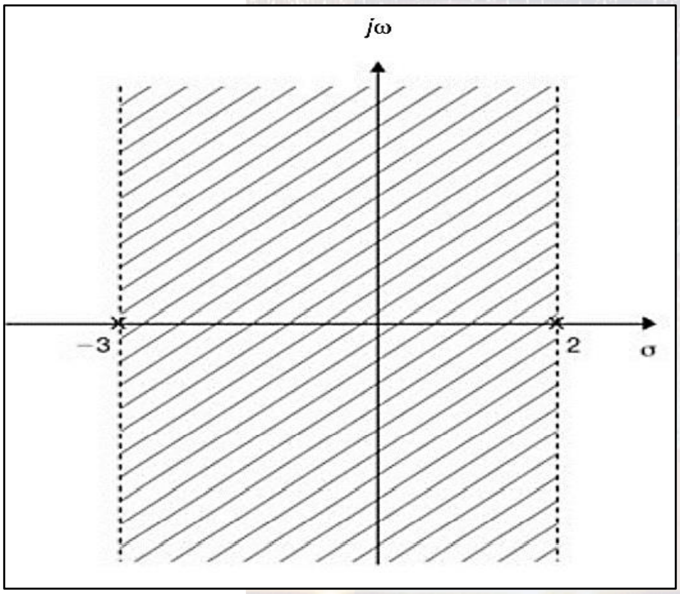


Figure 3.6 (b)

PROPERTIES OF THE ROC

1) There can't be any poles in ROC.

A pole is defined as a point where H(s) is infinite. H(s) must be finite in order to converge. As a result, there can't be a pole in ROC. The values for s where D(s) = 0 are called poles.If *x(t)* is absolutely integral and it is of finite duration, then ROC is entire *s-plane*

3) If *x(t)* is a right sided sequence, then ROC: $Re\{s\} > \sigma_o$

4) If *x(t)* is a left sided sequence, then ROC: $Re\{s\} < \sigma_o$

5) If *x(t)* is a two-sided sequence, then The ROC is the result of combining two regions.

LTI SYSTEM CHARACTERIZATION

CAUSALITY:

For a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0$$

As h(t) is a right-sided signal, H(s) must have a right-sided ROC.

$$Re(s) > \sigma_{max}$$

In other words, the ROC is the s-plane region to the right of all of the system poles. In the same way, if the system is anticausal,

$$h(t) = 0 \qquad t > 0$$

and h(t) is left-sided. Thus, the ROC of H(s) must be of the form

$$Re(s) < \sigma_{min}$$

In other words, the ROC is the s-plane region to the left of all of the system poles.

STABILITY:

We stated that the stability of a continuous-time LTI system is BIBO (bounded-input/bounded-output) if and only if

$$\int_{-\infty}^{\infty} |h(t)| \ dt < \infty$$

CAUSAL AND STABLE SYSTEM:

Because the ROC is of the form Re(s) > max and the jw axis is included in the ROC, all the poles of H(s) must be in the left half of the s-plane; that is, they must all have negative real values if the system is both causal and stable.

Figure 3.7 summarises the characterizations of LTI systems.

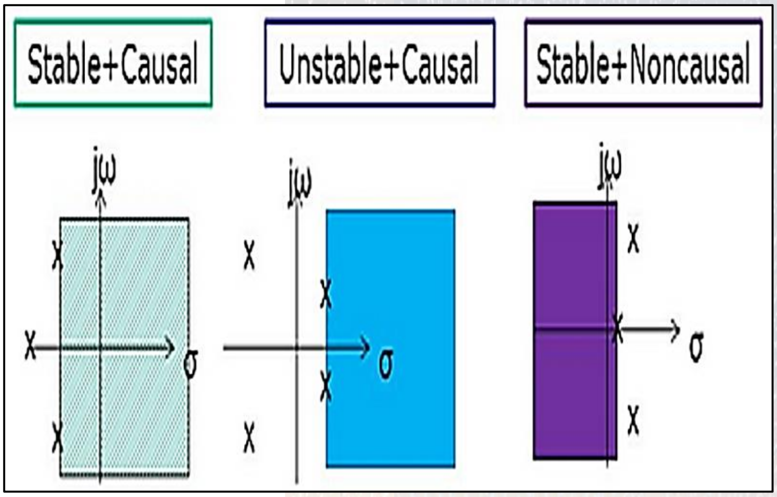


Figure 3.7
Laplace Causal and Stability Condition

PROPERTIES OF THE LAPLACE TRANSFORM

A. Linearity:

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$
$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$
$$a_1x_1(t) + a_2x_2(t) \leftrightarrow a_1X_1(s) + a_2X_2(s) \qquad R' \supset R_1 \cap R_2$$

The set notation A $\supset$ B means that set A contains set B, while A $\cap$ B denotes the intersection of sets A and B, that is, the set containing all value from both A and B. The ROC of the resulting Laplace transform is at least as large as the region in common between R1 and R2, according to this equation. Figure 3.8 shows how this performs.

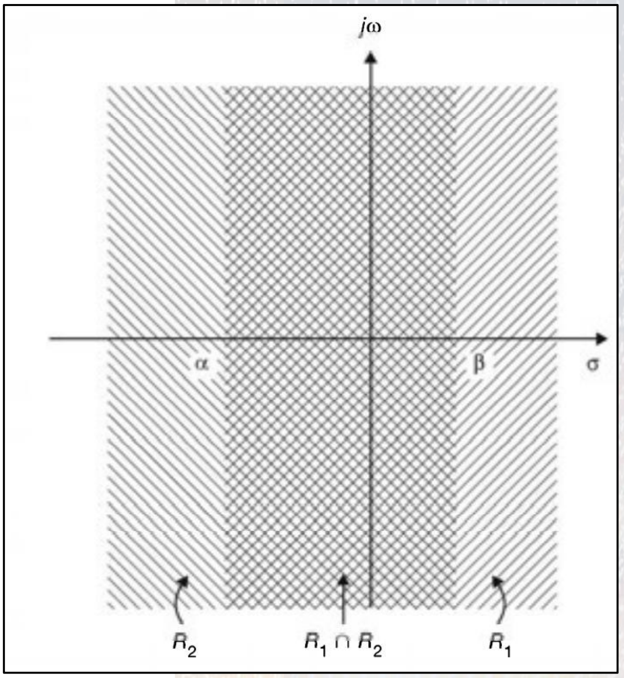


Figure 3.8
ROC of $a_1X_1(s) + a_2X_2(s)$

B. Time Shifting:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$
$$x(t - t_0) \leftrightarrow e^{-st_0} X(s) \qquad R' = R$$

C. Shifting in the s-Domain:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$
$$e^{s_0 t} x(t) \leftrightarrow X(s - s_0) \qquad R' = R + \text{Re}(s_0)$$

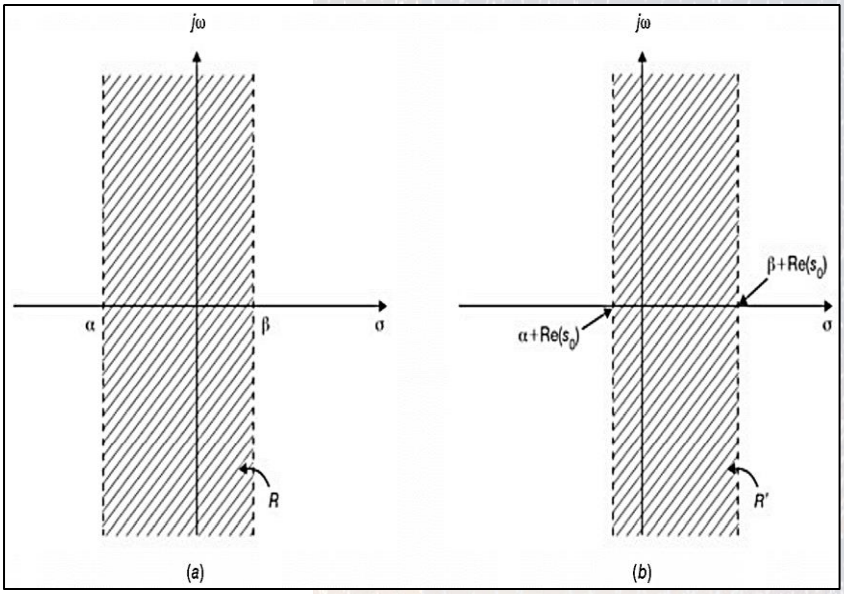


Figure 3.9
Shifting in the s-domain has an effect on the ROC.(a) ROC of X(s); (b) ROC of X (s - s_0).

D. Time Scaling:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right) \qquad R' = aR$$

Scaling the time variable t by the factor a results in an inverse scaling of the variable s by 1/a, as well as an amplitude scaling of X (s/a) by 1/|a|. Figure 3.10 shows the ROC effect of the corresponding effect.

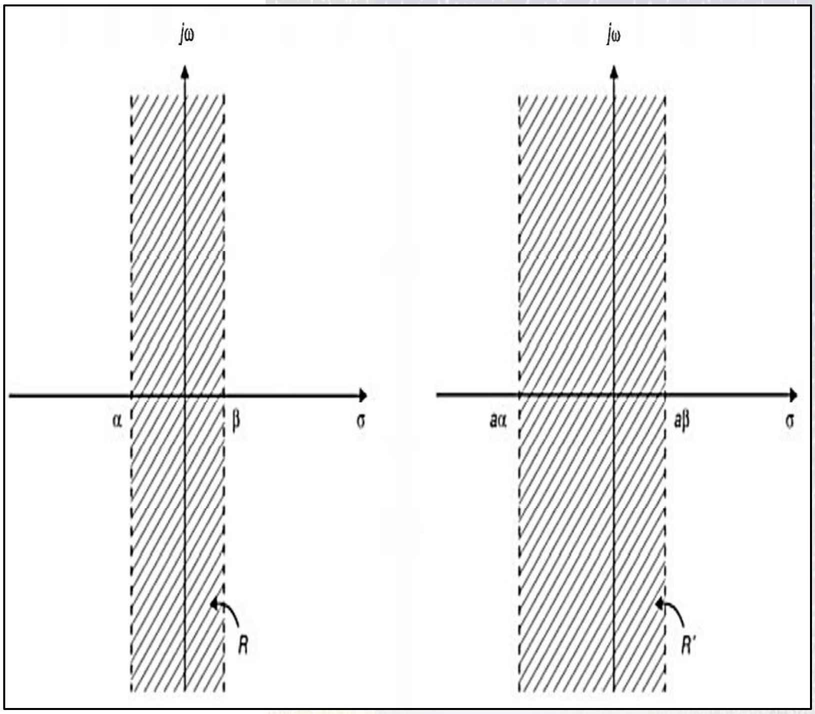


Figure 3.10
Shifting in the s-domain has an effect on the ROC (a) ROC of X(s); (b) ROC of X(s/a).

E. Time Reversal:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$
$$x(-t) \leftrightarrow X(-s) \qquad R' = -R$$

In the s-plane, time reversal of x(t) results in a reversal of both the a- and jw-axes. Setting a = -1 easily obtains the equation above.

F. Differentiation in the Time Domain:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad R' \supset R$$

G. Differentiation in the s-Domain:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad R' = R$$

H. Integration in the Time Domain:

$$x(t) \leftrightarrow X(s) \qquad \text{ROC} = R$$

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad R' = R \cap \{\text{Re}(s) > 0\}$$

If integration is the inverse operation of differentiation, the Laplace transform operation corresponding to time-domain integration is multiplied by 1/s, as can be seen in the equation above. The form of R' follows from the possible introduction of an additional pole at s = 0 by the multiplication by 1/ s.

I. Convolution:

$$x_1(t) \leftrightarrow X_1(s) \qquad \text{ROC} = R_1$$

$$x_2(t) \leftrightarrow X_2(s) \qquad \text{ROC} = R_2$$

$$x_1(t) * x_2(t) \leftrightarrow X_1(s)X_2(s) \qquad R' \supset R_1 \cap R_2$$

The features of the Laplace transform described in this section are summarised in Table 3-2.

Table 3-2 The Laplace Transform attributes

PROPERTY	SIGNAL	TRANSFORM	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$	$R' \supset R_1 \cap R_2$
Time shifting	$x(t-t_0)$	$e^{-st_0}\ X(s)$	$R' = R$
Shifting in s	$e^{s_0t}\ x(t)$	$X(s-s_0)$	$R' = R + \text{Re}(s_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X(a)$	$R' = aR$
Time reversal	$x(-t)$	$X(-s)$	$R' = -R$
Differentiation in t	$\frac{dx(t)}{dt}$	$sX(s)$	$R' \supset R$
Differentiation in s	$-tx(t)$	$\frac{dX(s)}{ds}$	$R' = R$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	$R' \supset R \cap \{\text{Re}(s) > 0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)\,X_2(s)$	$R' \supset R_1 \cap R_2$

Example 3.5

In Laplace, rewrite the following signals:

a. $x(t) = u(t-5)$

b. $x(t) = e^{5t}u(-t+3)$

Solution

a. $x(t) = u(t-5)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} e^{-st}u(t-5)dt \quad = \int_{5}^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}[e^{-st}]_5^{\infty} = -\frac{1}{s}[-e^{-5s}]$$

We obtain, $$= \frac{e^{-5s}}{s} \qquad R_e > 0$$

b. $x(t) = e^{5t}u(-t+3)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{\infty} e^{5t}u(-t+3)e^{-st}dt$$

$$= \int_{-\infty}^{3} e^{-(s-5)t}et \quad = -\frac{1}{s-5}\left[e^{-(s-5)t}\right]_{-\infty}^{3}$$

We obtain, $= -\frac{1}{s-5}e^{-3(s-5)} \qquad Re(s) < 5$

INVERSE LAPLACE TRANSFORM

Inversion of the Laplace transform to find the signal x(t) from its Laplace transform X(s) is called the inverse Laplace transform, symbolically denoted as

$$x(t) = L^{-1}\{x(s)\}$$

Example 3.6

Identified the inverse Laplace transform of the following X(s).

a) $X(s) = \frac{1}{s+1}, \quad \text{Re}(s) > -1$

b) $X(s) = \frac{1}{s+1}, \quad \text{Re}(s) < -1$

c) $X(s) = \frac{1}{s+1}, \quad \text{Re}(s) > 0$

d) $X(s) = \frac{s+1}{(s+1)^2+4}, \text{Re}(s) > -1$

Solution

From Table 3-1, we find:

$a)\ x(t) = e^{-t}u(t)$

$b)\ x(t) = -e^{-t}u(-t)$

$c)\ x(t) = \cos 2t\ \ u(t)$

$d)\ x(t) = e^{-t}\cos 2t\ \ u(t)$

Example 3.7

Solve the inverse Laplace transform of the following X(s):

$$(a) \quad X(s) = \frac{2s+4}{s^2+4s+3}, \ \text{Re}(s) > -1$$

$$(b) \quad X(s) = \frac{2s+4}{s^2+4s+3}, \ \text{Re}(s) < -3$$

$$(c) \quad X(s) = \frac{2s+4}{s^2+4s+3}, \ -3 < \text{Re}(s) < -1$$

Solution

a) $X(s) = \frac{2s+4}{s^2+4s+3} = \frac{2(s+2)}{(s+1)(s+3)} \qquad \mathrm{Re}(s) > -1$

Using partial fraction,

$$= \frac{A}{s+1} + \frac{B}{s+3} \quad = A(s+3) + B(s+1)$$

Replace $s = -1$

$A(s+3) = 2(s+2)$

$A(-1+3) = 2(-1+2)$

$2A = 2$

$A = 1$

Replace s= -3

$B(s+1) = 2(s+2)$

$B(-3+1) = 2(-3+2)$

$-2B = -2$

$B = 1$

$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}$$

$\mathrm{Re}(s) > -1$, x(t) from Table 3 − 1, we obtain

$$x(t) = e^{-t}u(t) + e^{-3t}u(t)$$

b) $\mathrm{Re}(s) < -3$

$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t)$

c) $-3 < \mathrm{Re}(s) < -1$

$x(t) = -e^{-t}u(-t) + e^{-3t}u(t)$

Example 3.8

Solve the inverse Laplace Transform of:

$$X(s) = \frac{s^2+6s+7}{s^2+3s+2}, \text{ Re(s) > -1}$$

Solution

Using long division:

$$\begin{array}{r|l} & 1 \\ s^2+3s+2 & s^2+6s+7 \\ & -\, s^2+3s+2 \\ \hline & \quad 3s+5 \end{array}$$

$$X(s) = 1 + \frac{3s+5}{s^2+3s+2}$$

$$\frac{3s+5}{s^2+3s+2} = \frac{3s+5}{(s+1)(s+2)}$$

Using partial fraction,

$$\frac{A}{s+1} + \frac{B}{s+2} = A(s+2) + B(s+1)$$

Replace s= -1,

$$3s + 5 = A(s+2)$$

$$3(-1) + 5 = A(-1+2)$$
$$2 = A$$

Replace s = -2

$$3S + 5 = B(s+1)$$

$$3(-2) + 5 = 13(-2+1)$$
$$-1 = -B$$
$$B = 1$$

$$X(s) = 1 + \frac{2}{s+1} + \frac{1}{s+2}$$

From Table 3-1, we obtain:

$$x(t) = \delta(t) + 2e^{-t}u(t) + e^{-2t}u(t)$$

Example 3.9

Calculate the inverse Laplace transform of:

$$L^{-1}\left(\frac{s}{s^2+2s+5}\right)$$

Solution

We solve the squared to produce a shifted sine or cosine form because the denominator expression cannot be factored into linear terms.

$$L^{-1}\left(\frac{s}{s^2+2s+5}\right) = \frac{s}{s^2+2s+\mathbf{1}-\mathbf{1}+5}$$

$$= \frac{s}{[s^2+2s+1]-1+5}$$

$$= \frac{s}{(s+1)^2+4}$$

Replace s= s+1,

$$= \frac{(s+\mathbf{1})-\mathbf{1}}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+2^2} - \frac{\mathbf{1}}{\mathbf{2}} \times \frac{\mathbf{2}}{(s+1)^2+2^2}$$

From Table 3-1, we obtain:

$$= e^{-t}\cos 2t\, u(t) - \frac{1}{2}e^{-t}\sin 2t\, u(t)$$

SYSTEM FUNCTION

A continuous-time LTI system's output y(t) is equal to the convolution of the input x(t) with the impulse response h(t); that is,

$$y(t) = x(t) * h(t)$$

We get the following result by using the convolution property.

$$Y(s) = X(s) + H(s)$$

where Y(s), X(s), and H(s) are the Laplace transforms of y(t), x(t), and h(t), respectively.

The system function (or the transfer function) of the system is the Laplace transform H(s) of h(t).

$$H(s) = \frac{Y(s)}{X(s)}$$

The ratio between the Laplace transforms of the output y(t) and the input x can also be defined as the system function H(s) (t). Because the impulse response h(t) entirely characterizes the system, the system function H(s) completely classifies the system. The connection H(s) = Y(s)X(s) is depicted in Figure 3.11.

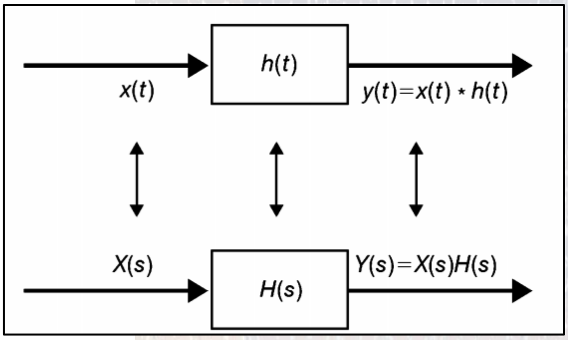
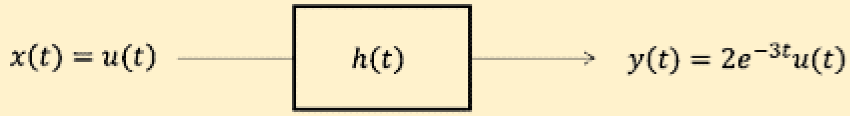


Figure 3.11
System function and impulse response.

Example 3.9



A continuous-time LTI system's output y(t) is discovered to be $2e^{-3t}u(t)$ where x(t) is equal to u (t).

a) Find the input response h(t) of the system.
b) Find the output y(t) when the input x(t) is $e^{-t}u(t)$.

Solution

$a) \quad y(t) = 2e^{-3t}u(t)$

Using Table 3-1;

$$Y(s) = 2\frac{1}{s+3} \qquad Re(s) > -3$$

$$x(t) = u(t)$$

Using Table 3-1,

$X(s) = \frac{1}{s}$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{2}{s+3}}{\frac{1}{s}}$$

$$H(s) = \frac{2}{s+3} \times s = \frac{2s}{s+3}$$

Using long division;

$$H(s) = 2 - \frac{6}{s+3}$$

From Table 3-1, we obtain;

$$h(t) = 2\delta(t) - 6e^{-3t}u(t)$$

$b)$ $x(t) = e^{-t}u(t)$

Using Table 3-1;

$$x(s) = \frac{1}{s+1} Re(s) > -1$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s) \ x \ X(s)$$

$$\mathrm{Y}(s) = \frac{2s}{s+3} \times \frac{1}{s+1}$$

$$= \frac{2s+3}{(s+3)(s+1)}$$

Using partial fraction;

$$2s = A(s+1) + B(s+3)$$

Replace s= -1

$$2(-1) = B(2)$$
$$B = -1$$

Replace s = -3

$$2(-3) = A(-2)$$
$$A = 3$$

$$Y(s) = \frac{3}{s+3} - \frac{1}{s+1}$$

From Table 3-1, we find:

$$y(t) = 3e^{-3t}u(t) - e^{-t}u(t)$$

Example 3.10

Consider a continuous time LTI with a relation between input x(t) and output y(t).

$$y''(t) + y'(t) - 2(y) = x(t)$$

a) Solve the transfer function of the system, H(s).
b) Identified the impulse response h(t) for each of the following three cases:
   i. The system is causal
   ii. The system is stable
   iii. The system is neither causal or nor stable

Solution

$$y''(t) + y'(t) - 2(y) = x(t)$$

Taking the Laplace transform of the question 3.10, we have

$$s^2Y(s) + sY(s) - 2Y(s) = \mathrm{X}\,(s)$$

$$(s^2 + s - 2)y(s) = x(s)$$

a) $H(s) = \frac{Y(s)}{X(s)}$

$= \frac{1}{S^2+S-2}$

the answer: $= \frac{1}{(s-1)(s+2)}$

b) $H(s) = \frac{1}{(s-1)(s+2)}$

$$1 = A(s+2) + B(s-1)$$

Replace s = -2

$$1 = B(-2-1)$$
$$B = -\frac{1}{3}$$

Replace s = 1

$$1 = A(3)$$
$$A = \frac{1}{3}$$

$$H(s) = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}$$

$$h(t) = \frac{1}{3}\left(e^{t} - e^{-2t}\right)$$

i) System causal Re(s)>1, we find

$$h(t) = \frac{1}{3}e^{t}u(t) - \frac{1}{3}e^{-2t}u(t)$$

ii) System causal -2< Re(s)<1 , we find

$$h(t) = -\frac{1}{3}e^{t}u(-t) - \frac{1}{3}e^{-2t}u(t)$$

ii) Neither causal or nor stable Re(s)<-2 , we find

$$h(t) = -\frac{1}{3}e^{t}u(-t) + \frac{1}{3}e^{-2t}u(-t)$$

Supplementary Problems

1. Calculate the Laplace transform of the following x(t)
   a) $x(t) = \sin \omega_0 t u(t)$
   b) $x(t) = \cos(\omega_0 t + \Phi)u(t)$
   c) $X(t) = e^{-at}u(t) - e^{at}u(-t)$

2. Solve Inverse transform function of

$$F(s) = \frac{6s^2 + 10s + 2}{s^3 + 3s^2 + 2s}$$

3. Solve the inverse Laplace transform of:

$$L^{-1}\left(\frac{s+1}{s^2 - 6s + 13}\right)$$

4. A continuous-time LTI system's step response is provided by $(1 - e^{-t})\, u(t)$. The result y(t) is observed to be for a given unknown input x(t). $(2 - 3e^{-t}, + e^{-3t})u(t)$. Find the input x(t).

SCAN ME

FOR ANSWER

Dream big, stay positive, work hard, and enjoy the journey

~Urijah Faber~

TOPIC 2

Z Transform

INTRODUCTION

The z-transform is the most basic concept for discrete-time series transformation.

The Laplace transform is a more general concept for continuous time process transformation.

The Laplace transform, for example, can be used to convert a differential equation and its associated initial and boundary value problems into a space where the equation can be solved using standard algebra.

Operational calculus is the process of switching spaces to convert calculus equations into algebraic operations on transformations. The most essential methods for this are the Laplace and z transforms.

DEFINITION

A sequence of x[n] has a Z-transform defined by :

- Two-sided z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- One-sided z-transform (for causal system):

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

LAPLACE VS Z TRANSFOM

	Definition	Purpose
Laplace transform	$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$	Integral-differential equations are converted to algebraic equations.
Z transform	$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$	Differential conditions are changed over to arithmetical conditions.

Example 3.11

Find the z-transform of these sequences:

$$x[n] = \{5,3,\underline{-2},0,4,-3\}$$

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$= \sum_{n=-2}^{3} x[n]z^{-n}$$

$$= x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3}$$

We obtain;

$$= 5z^{2} + 3z - 2 + 4z^{-2} - 3z^{-3}$$

Example 3.12

Take a look at the x[n] sequence in the table below.

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

We obtain;

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Z TRANSFORM PAIRS

The z-transforms of some common sequences are in Table 3.3.

Table 3.3 The z-transform pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if $(m>0)$ or ∞ if $(m<0)$
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left[\frac{z}{z-a}\right]^2$	$\lvert z\rvert > \lvert a\rvert$
$(\cos \Omega_0 n)u[n]$	$\frac{z^2-(\cos \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(\sin \Omega_0 n)u[n]$	$\frac{(\sin \Omega_0)z}{z^2-(2\cos \Omega_0)z+1}$	$\lvert z\rvert > 1$
$(r^n \cos \Omega_0 n)u[n]$	$\frac{z^2-(r\cos \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$(r^n \sin \Omega_0 n)u[n]$	$\frac{(r\sin \Omega_0)z}{z^2-(2r\cos \Omega_0)z+r^2}$	$\lvert z\rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Noted. Adapted from Signal and System, by Hwei, 2010, p. 141. The McGraw-Hill Companies, Inc. owns the copyright to this work.

Example 3.13

Calculate the z-transform of:

a) $x[n] = a^n u[n]$
b) $x[n] = u[n-1]$
c) $x[n] = -a^n u[-n-1]$
d) $x[n] = a^{-n} u[-n-1]$

Solution

a) $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

Using Summation Formula;

$$= \frac{1}{1-az^{-1}} x \frac{z}{z}$$

$$= \frac{z}{z-a}$$

Solution

b) $x[n] = u[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} u[n-1] z^{-n}$$

$$= \sum_{n=1}^{\infty} (z^{-1})^n$$

Using Summation Formula;

$$= \frac{z^{-1}}{1-z^{-1}} x \frac{z}{z}$$

$$= \frac{1}{z-1}$$

Solution

c) $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1]z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -(-1) - \sum_{n=0}^{\infty} a^{-n} z^{n}$$

Change polarity

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{a^{-1}z}$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} \times \frac{-a}{-a}$$

We get;

$$= \frac{z}{z-a}$$

Solution

d) $x[n] = a^{-n} u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[-n-1]z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n}$$

Change polarity

$$= (-1) + \sum_{n=0}^{\infty} a^n z^n$$

$$= \sum_{n=0}^{\infty} (az)^n - 1 = \frac{1}{1-az} - 1$$

$$= \frac{1-1+az}{1-az}$$

$$= \frac{az}{1-az} \times \frac{\frac{1}{a}}{\frac{1}{a}}$$

We get; $$= \frac{z}{\frac{1}{a} - z} = -\frac{z}{z - \frac{1}{a}}$$

REGION OF CONVERGENCE

Convergence Region The ROC is significant since it determines the area in which the Z-Transform can be found. The z-transform should be represented as a rational function.

$$X(z) = \frac{P(z)}{Q(z)}$$

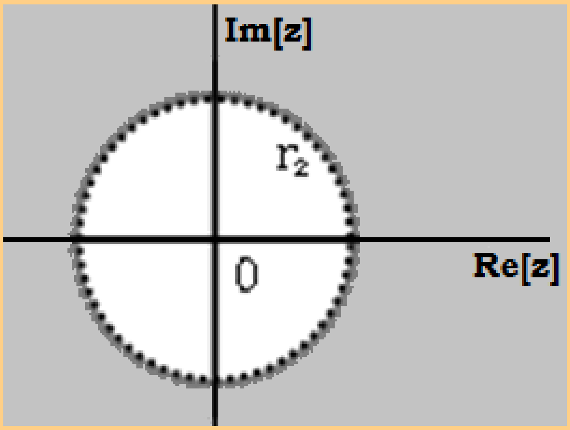
$P(z)$ → Zeros

$Q(z)$ → Poles

where $P(z)$ and $Q(z)$ are polynomials in z.

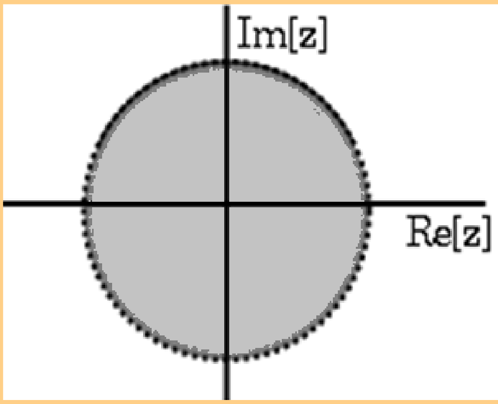
PROPERTIES OF THE ROC

1) In the z-plane, a ring or disc centred at the origin.
2) ROC cannot contain any poles .
3) Finite duration sequences : The ROC is the entire z-plane except possibly z=0 or z= ∞
4) **Right sided sequences (Causal):** From the outermost finite pole in X(z) to z = ∞, the ROC spreads outward.



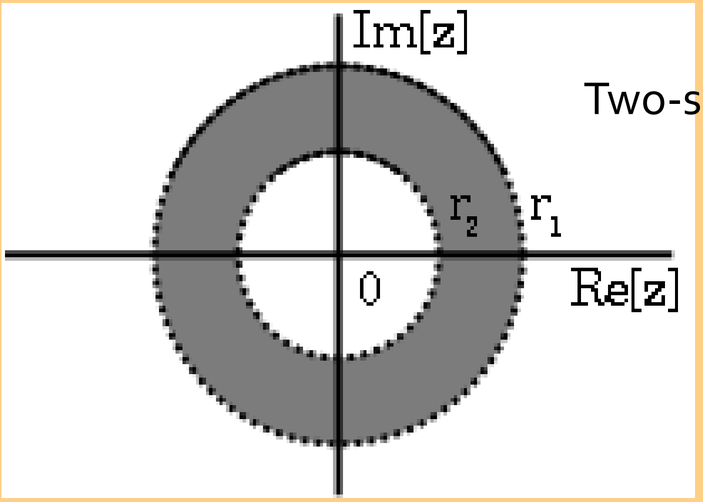
Causal: x(n) =0 and n < 0

5) **Left sided sequences (anti-Causal):** From the innermost nonzero pole in X(z) to z = 0, the ROC extends inward.



AntiCausal: x(n) =0 and n >0

6) Two-sided sequence : The ROC is a ring with no poles inside it, bounded by two circles passing through two poles.



Two-sided = Anticausal + Causal

Example 3.14

Calculate the z-transform *X(z)* and sketch the pole-zero plot with ROC for

1) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

2) $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$

Solution

From Table 3.3;

$$\left(\frac{1}{2}\right)^n u[n] \rightarrow \frac{z}{2 - \frac{1}{2}} \qquad |z| > \frac{1}{2}$$

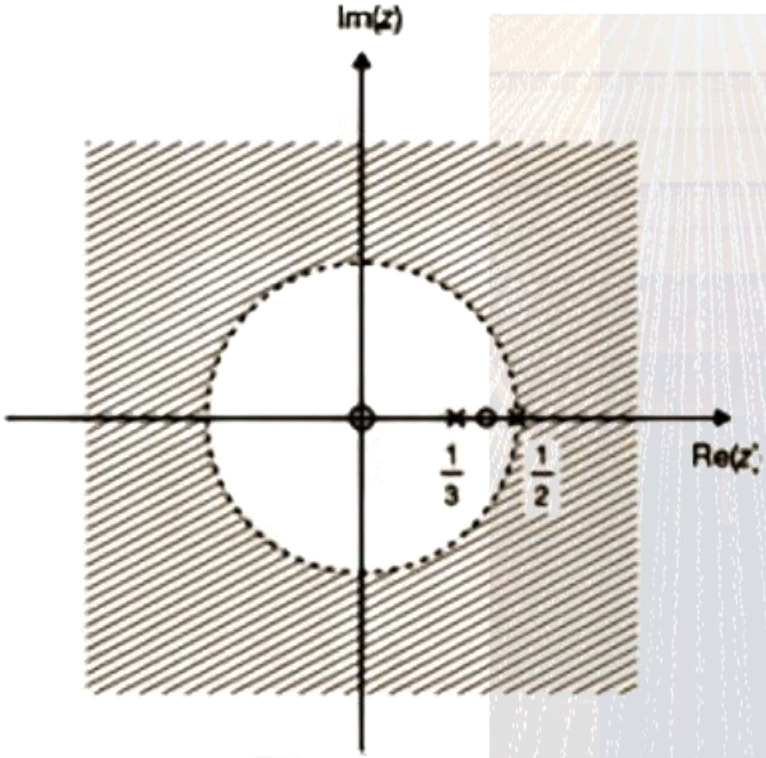
$$\left(\frac{1}{3}\right)^n u[n] \rightarrow \frac{z}{z - \frac{1}{3}} \qquad |z| > \frac{1}{3}$$

1) $$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}} = \frac{z\left(z-\frac{1}{3}\right)+z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{z^2 - \frac{1}{3}z + z^2 - \frac{1}{2}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{2z^2 - \frac{5}{6}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{2z\left(z-\frac{5}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

We see has two zero : z=0, z=5/12 and two poles z=1/2, z=1/3 . That ROC is |z| $> \frac{1}{2}$, as shown in figure below



Solution

From Table 3.3;

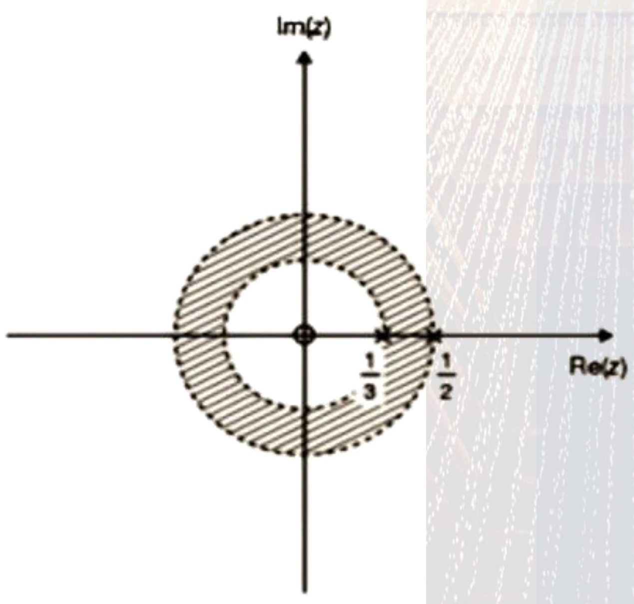
$$\left(\frac{1}{3}\right)^n u[n] \rightarrow \frac{z}{2-\frac{1}{3}} \qquad |\mathrm{z}| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[-n-1] \rightarrow -\frac{z}{z-\frac{1}{2}} \qquad |z| < \frac{1}{2}$$

$$x[n] = \frac{z}{z-\frac{1}{3}} - \frac{z}{z-\frac{1}{2}} = \frac{z\left(\mathrm{z}-\frac{1}{2}\right) - z\left(z-\frac{1}{3}\right)}{\left(\mathrm{z}-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$= \frac{-\frac{1}{6}z}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

We see has one zero : z=0 and two poles z=1/2, z=1/3 . That ROC is $\frac{1}{3} < |\mathrm{z}| < \frac{1}{2}$, as plotted in figure below



INVERSE Z TRANSFORM

The inverse z-transform is the inversion of the z-transform to get the sequence x[n] from its z-transform X(z).

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

An inverse Z Transform can be done in numerous ways:

- Partial fraction expansion
- Long division
- Power series expansion

Example 3.14

Find the inverse z transform of the following signal using the power series expansion technique.

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < \frac{1}{2}$$

Solution

Since the ROC is $|z| < \frac{1}{2}$, x[n] is a left-sided sequence

$$X(z) = \frac{z}{2z^2 - 3z + 1} \qquad |z| < \frac{1}{2}$$

$$\begin{array}{r l}
 & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
1 - 3z + 2z^2 \,\big) & z \\
 & \underline{z - 3z^2 + 2z^3} \\
 & 3z^2 - 2z^3 \\
 & \underline{3z^2 - 9z^3 + 6z^4} \\
 & 7z^3 - 6z^4 \\
 & \underline{7z^3 - 21z^4 + 14z^5} \\
 & 15z^4 \cdots
\end{array}$$

Thus,

$$x(z) = \cdots + 15z^4 + 7z^3 + 3z^2 + z$$

And so by definition we obtain

$$x[n] = \{\cdots, 15,7,3,1,0\}$$

Example 3.15

Find the inverse z transform of the following signal using the partial fraction expansion technique.

$$\mathrm{X}(z) = \frac{z}{z(z-1)(z-2)^2} \qquad |z| > 2$$

Solution

Using partial-fraction expansion,

$$x(z) = \frac{z}{z(z-1)(z-2)^2} \quad x\,\frac{1}{z} \qquad |z| > 2$$

$$\frac{x(z)}{z} = \frac{1}{z(z-1)(2-2)^2}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$1 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

$$1 = A(z^2 - 4z + 4) + B(z^2 - 3z + 2) + C(z-1)$$

Replace z=1;

$$1 = A(1 - 4 + 4) + B(1 - 3 + 2)$$
$$1 = A$$

Replace z=2;

$$1 = B(4 - 6 + 2) + C(2 - 1)$$
$$1 = C$$

Solve z^2;

$$A + B = 0$$
$$1 + B = 0$$
$$B = -1$$

Thus

$$\left\{\frac{x(z)}{z} = \left[\frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2}\right]\right\} \times z$$

$$x(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2}$$

Since the ROC is I z I > 2, x[n] is a right-sided sequence, and from Table 3-3 we get

$$x[n] = u[n] - 2^n u[n] + n2^{n-1}u[n]$$

Example 3.16

Solve inverse z transform of the following signal

$$x(z) = \frac{1-z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})(1-z^{-1})} \quad , \qquad 1 < z < 2$$

Solution

$$x(z) = \frac{1-z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})(1-z^{-1})} \text{ x } \frac{z^3}{z^3}$$

$$X(z) = \frac{z^3 - z^2 + 1}{\left(z-\frac{1}{2}\right)(z-2)(z-1)} \, x \, \frac{1}{z}$$

$$\frac{X(z)}{z} = \frac{z^2 - z^1 + 2}{\left(z-\frac{1}{2}\right)(z-2)(z-1)}$$

$$z^2 - z + 1 = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-2} + \frac{C}{z-1}$$

We obtain,

$$A = 1, B = 2, C = -2$$

Thus,

$$\left\{\frac{x(z)}{z} = \left[\frac{1}{z-\frac{1}{2}} - \frac{2}{z-2} + \frac{2}{z-1}\right]\right\} \times z$$

$$x(z) = \frac{z}{z-\frac{1}{2}} - \frac{2z}{z-2} + \frac{2z}{z-1}$$

Since the ROC is 1< I z I > 2, x[n] is a right-sided sequence, and from Table 3-3 we get

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - 2(2)^n u[-n-1] - 2u[n]$$

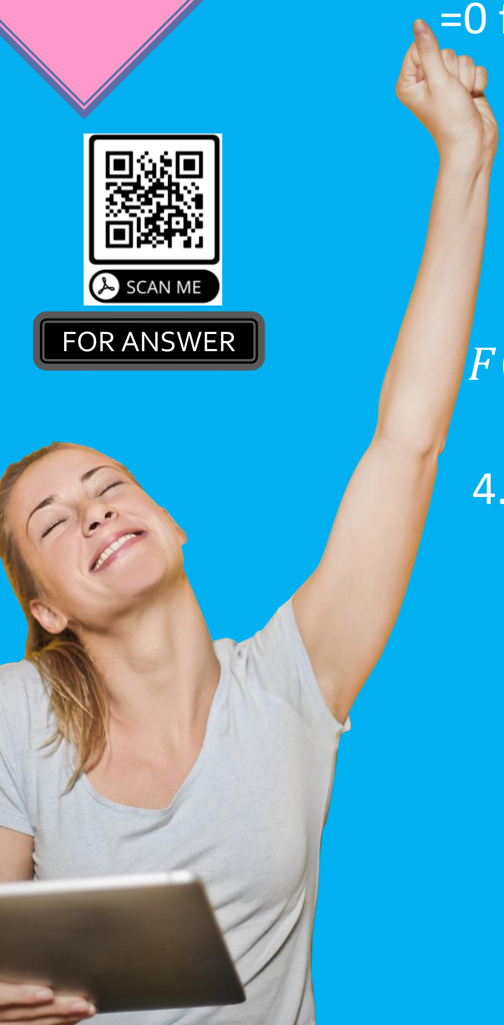
Supplementary Problems

No pressure, no diamonds

~ Thomas Carlyle ~

SCAN ME

FOR ANSWER



1. Transform the x[n] signal into an x-domain signal.

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & otherwise \end{cases}$$

2. When x[n] is known to be causal, find x[n] from X(z) below using partial fraction expansion. X[n] =0 for n<0

$$x(z) = \frac{3 + 2z^{-1}}{2 + 3z^{-1} + 2z^{-2}}$$

3. Solve the inverse z transform of the following signal

$$F(z) = \frac{1}{(1 - 0 \cdot 5z^{-1})(1 - 0.75z^{-1})(1 - z^{-1})}$$

4. Consider a discrete-time system with a causal relationship between the output y[n] and the input x[n].

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n]$$

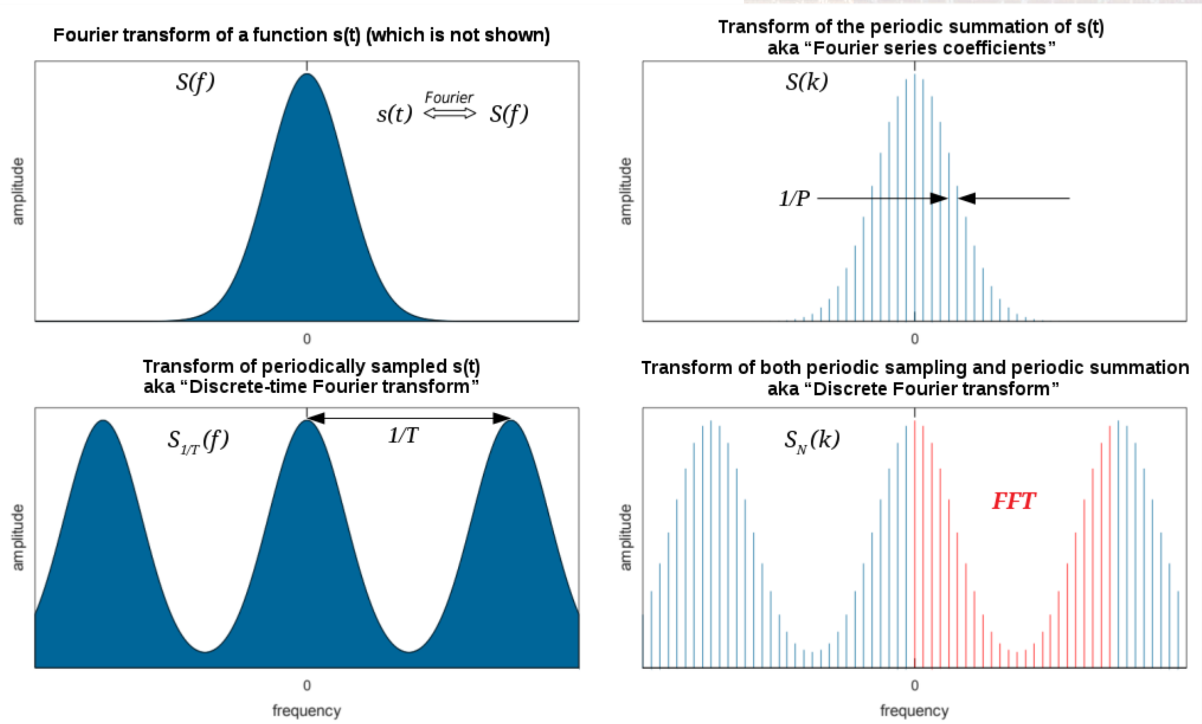
   a) Find its system function H(z).
   b) Calculate its impulse response h[n].

TOPIC 3

FOURIER ANALYSIS OF CONTINUOUS-TIME AND DISCRETE-TIME SIGNALS & SYSTEMS

INTRODUCTION

Understanding the behavior of signals and systems requires the use of Fourier analysis. Sinusoids are Eigenfunctions of linear, time-invariant (LTI) systems, hence this is the case. This means that when we run a sinusoid through an LTI system, we get a scaled version of the same sinusoid on the output.



We can redefine signals in terms of sinusoids using Fourier analysis; all we have to do is figure out how any given system affects all available sinusoids and we have a comprehensive knowledge of the system. We may also transform the passage of any signal through a system from convolution (in time) to multiplication (at the same frequency) since we can define the passage of sinusoids through a system as multiplication of that sinusoid by the transfer function at the same frequency (in frequency).

PERIODIC SIGNAL

A continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \text{ for all } t$$

The fundamental period T_o of $x(t)$ is the smaller positive value of T and $1/T_o = f_o$ is referred to as the *fundamental frequency*

Real sinusoidal signal

$$x(t) = \cos(\omega_o t + \phi)$$

The complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

Fundamental angular frequency

$$\Omega_o = 2\pi/T_o = 2\pi f_o$$

"Fear and pain should be treated as signals not to close our eyes but to open them wider"
~Nathaniel Branden~

COMPLEX EXPONENTIAL FOURIER SERIES

The complex exponential Fourier series representation of a periodic signal *x(t)* with fundamental period *To* is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

EXPONENTIAL FOURIER SERIES

The complex Fourier coefficients, Ck, are defined as follows:

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

Where ∫To denotes the integral over any one period and 0 to To or –To/2 to To/2 is commonly used for the integration. If k=0

$$c_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

Which means Co is equal to the average value of x(t) across time.

Example 4.1

Determine the complex exponential Fourier series representation for each of the following signals:

(a) $x(t) = \cos \omega_0 t$

(b) $x(t) = \sin \omega_0 t$

Solution

To evaluate the complex Fourier Coefficient, *Ck*, using Euler's formula,

$$\cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier Coefficient for $\cos \omega_0 t$

$$c_1 = \frac{1}{2} \qquad c_{-1} = \frac{1}{2} \qquad c_k = 0, |k| \neq 1$$

In similar

$$\sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) = -\frac{1}{2j}e^{-j\omega_0 t} + \frac{1}{2j}e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Thus, the complex Fourier Coefficient for $\sin \omega_0 t$

$$c_1 = \frac{1}{2j} \qquad c_{-1} = -\frac{1}{2j} \qquad c_k = 0, |k| \neq 1$$

TRIGONOMETRIC FOURIER SERIES (FS)

A periodic signal x(t) with fundamental period To has a trigonometric Fourier series representation given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}$$

Where a_k and b_k are the Fourier coefficients given by

$$a_k = \frac{2}{T_0} \int_{T_0} x(t) \cos k\omega_0 t \, dt$$

ODD SIGNAL

$$b_k = \frac{2}{T_0} \int_{T_0} x(t) \sin k\omega_0 t \, dt$$

EVEN SIGNAL

If x(t) is an even periodic signal, then $b_k = 0$ and the Fourier series contains just the cosine term.

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

If x(t) is an even periodic signal, then $a_k = 0$ and the Fourier series contains solely sine terms.

$$x(t) = \sum_{k=1}^{\infty} b_k \sin k\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

DISCRETE FOURIER SERIES

A discrete time signal *x[n]* to be periodic if there is a positive integer N

$$x[n+N_0] = x[n]$$

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

where C_k are the Fourier coefficients and given by

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$$

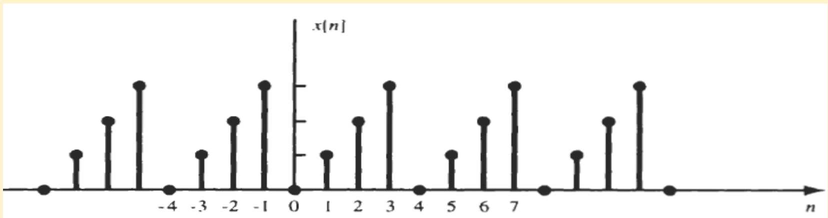
$$c_{-k} = c_{N_0-k} = c_k^*$$

$$j^2 = -1$$

What is real? How do you define real? If you're talking about what you can feel, what you can smell, what you can taste and see, then real is simply electrical signals interpreted by your brain.
~Lana Wachowski~

Example 4.2

Determine the Fourier coefficient for the periodic sequence *x[n]* shown in figure below.



Solution

The periodic extension of {0,1,2,3} with fundamental period No = 4

$$\Omega_0 = \frac{2\pi}{4} \quad \text{and} \quad e^{-j\Omega_0} = e^{-j2\pi/4} = e^{-j\pi/2} = -j$$

The discrete time Fourier coefficient C_k are

$$c_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{4}(0+1+2+3) = \frac{3}{2}$$

$$c_1 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^n = \frac{1}{4}(0-j1-2+j3) = -\frac{1}{2}+j\frac{1}{2}$$

$$c_2 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{2n} = \frac{1}{4}(0-1+2-3) = -\frac{1}{2}$$

$$c_3 = \frac{1}{4}\sum_{n=0}^{3} x[n](-j)^{3n} = \frac{1}{4}(0+j1-2-j3) = -\frac{1}{2}-j\frac{1}{2}$$

Example 4.3

Consider the following sequence.

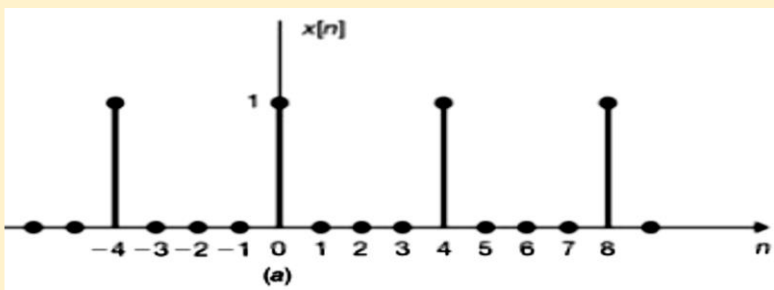
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$$

a) Sketch *x[n]*

b) Find the Fourier Coefficients, *Ck* of *x[n]*

Solution

a) *x[n]* is the periodic extension of the sequence {1,0,0,0} with period *No – 4*.

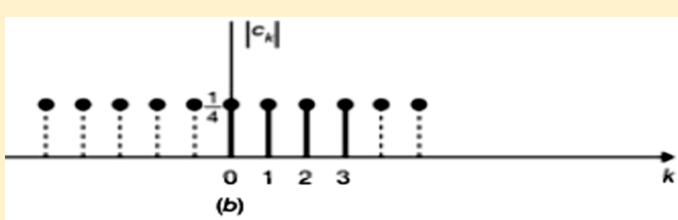


b)

$$x[n] = \sum_{k=0}^{3} c_k\, e^{jk(2\pi/4)n} = \sum_{k=0}^{3} c_k\, e^{jk(\pi/2)n}$$

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk(2\pi/4)n} = \frac{1}{4}x[0] = \frac{1}{4} \qquad \text{all } k$$

Since *x[1] = x[2] = x[3] = 0*, the Fourier coefficient of *x[n]* as shown in Figure below:



Example 4.4

Determine the discrete Fourier series representation for each of the following sequences:

$$x[n] = \cos\frac{\pi}{4}n$$

Solution

The fundamental period of x[n] is No = 8, and $\Omega_o = 2\pi/N_o = \pi/4$.
By using Euler's formula, Fourier coefficients C_k

$$\cos\frac{\pi}{4}n = \frac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$$

Thus, the Fourier coefficients for x[n] are $C_1 = ½$, $C_{-1} = C_{-1+8} = C_7 = ½$, and all other $C_k = 0$.

Hence, the discrete Fourier series of x[n] is

$$x[n] = \cos\frac{\pi}{4}n = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$$

FOURIER TRANSFORM

The Fourier transform is a mathematical approach for decomposing a Magnetic Resonance signal into a sum of sine waves with various frequencies, phases, and amplitudes.

A mathematical procedure converting a signal s(t) in the time domain to a complex number S(ω) in the frequency domain

Any periodic signal s(t) may be expressed as a sum of sine waves of variable amplitudes, frequencies, and phases, according to Fourier.

$$s(t) = a_0 + a_1 \sin(\omega t + \phi_1) + a_2 \sin(2\omega t + \phi_2) + a_3 \sin(3\omega t + \phi_3) + \cdots$$

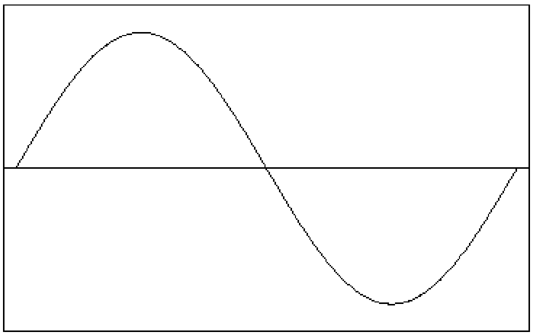
If the amplitudes are ϕ, the phase shifts are is, and the fundamental frequency.

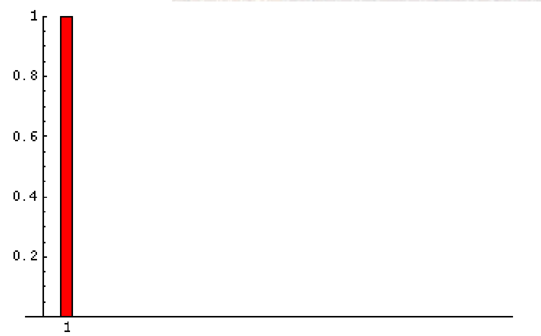
Harmonics are higher order frequencies such as 2, 3, and so on.

The Fourier expansion of a square wave, for example, may be represented as

$$s(t) = \sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \cdots$$

FOURIER TRANSFORM

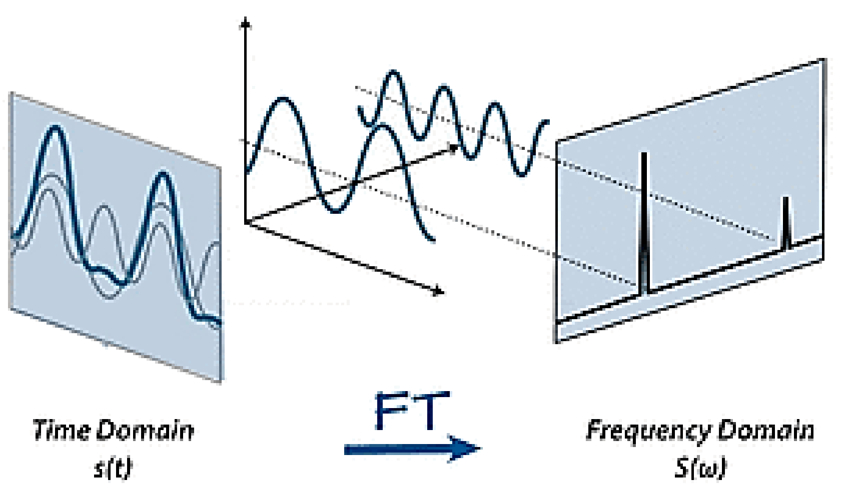




On the left, the square wave's time domain signal, s(t), is depicted.

On the right, the so-called frequency domain representation, S(ω), is presented. The Fourier transform of s(t) is known as S(ω).

S(ω) is a complex-valued function that is made up of harmonic frequencies, phases, and amplitudes derived via the Fourier expansion.



$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt$$

Fourier Transform

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} d\omega$$

Inverse Fourier Transform

FOURIER TRANSFORM

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

FOURIER TRANSFORM VS LAPLACE TRANSFORM

FOURIER TRANSFORM	LAPLACE TRANSFORM
$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$	$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$

PROPERTIES OF FOURIER TRANSFORM

CT Signal

Properties of the Fourier Transform

PROPERTY	SIGNAL	FOURIER TRANSFORM
	$x(t)$	$X(\omega)$
	$x_1(t)$	$X_1(\omega)$
	$x_2(t)$	$X_2(\omega)$
Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Frequency shifting	$e^{j\omega_0 t} x(t)$	$X(\omega - \omega_0)$
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time reversal	$x(-t)$	$X(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time differentiation	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency differentiation	$(-jt)x(t)$	$\frac{dX(\omega)}{d\omega}$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\pi X(0)\,\delta(\omega) + \frac{1}{j\omega} X(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Real signal	$x(t) = x_e(t) + x_o(t)$	$X(\omega) = A(\omega) + jB(\omega)$ $X(-\omega) = X^*(\omega)$
Even component	$x_e(t)$	$\text{Re}\{X(\omega)\} = A(\omega)$
Odd component	$x_o(t)$	$j\,\text{Im}\{X(\omega)\} = jB(\omega)$
Parseval's relations		

$$\int_{-\infty}^{\infty} x_1(\lambda)X_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} X_1(\lambda)x_2(\lambda)\, d\lambda$$

$$\int_{-\infty}^{\infty} x_1(t)x_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)X_2(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

FOURIER TRANSFORM

CT Signal

Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega)-\frac{1}{j\omega}$
$e^{-at}u(t), a>0$	$\frac{1}{j\omega+a}$
$te^{-at}u(t), a>0$	$\frac{1}{(j\omega+a)^2}$
$e^{-a\|t\|}, a>0$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2}, a>0$	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$
$p_a(t)=\begin{cases}1 & \|t\|<a\\ 0 & \|t\|>a\end{cases}$	$2a\frac{\sin\omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega)=\begin{cases}1 & \|\omega\|<a\\ 0 & \|\omega\|>a\end{cases}$
$\operatorname{sgn} t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0), \omega_0=\frac{2\pi}{T}$

PROPERTIES OF FOURIER TRANSFORM DT Signal

Properties of the Fourier Transform

PROPERTY	SEQUENCE	FOURIER TRANSFORM
	$x[n]$	$X(\Omega)$
	$x_1[n]$	$X_1(\Omega)$
	$x_2[n]$	$X_2(\Omega)$
Periodicity	$x[n]$	$X(\Omega + 2\pi) = X(\Omega)$
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(\Omega) + a_2X_2(\Omega)$
Time shifting	$x[n - n_0]$	$e^{-j\Omega n0}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 n}x[n]$	$X(\Omega - \Omega_0)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n = km \\ 0 & \text{if } n \neq km \end{cases}$	$X(m\Omega)$
Frequency differentiation	$nx[n]$	$j\dfrac{dX(\Omega)}{d\Omega}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}}X(\Omega)$ $\lvert\Omega\rvert \leq \pi$
Convolution	$x_1[n] * x_2[n]$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1[n]x_2[n]$	$\dfrac{1}{2\pi}X_1(\Omega) \otimes X_2(\Omega)$
Real sequence	$x[n] = x_e[n] + x_o[n]$	$X(\Omega) = A(\Omega) + jB(\Omega)$ $X(-\Omega) = X^*(\Omega)$
Even component	$x_e[n]$	$\text{Re}\{X(\Omega)\} = A(\Omega)$
Odd component	$x_o[n]$	$j\,\text{Im}\{X(\Omega)\} = jB(\Omega)$
Parseval's theorem		

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega)X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(\Omega\rvert^2\, d\Omega$$

FOURIER TRANSFORM

DT Signal

Common Fourier Transform Pairs

$x[n]$	$X(\Omega)$
$\delta[n]$	1
$\delta(n-n_0)$	$e^{-j\Omega n_0}$
$x[n]=1$	$2\pi\delta(\Omega), \lvert\Omega\rvert \le \pi$
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0), \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\sin\Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)], \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$u[n]$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$-u[-n-1]$	$-\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}, \lvert\Omega\rvert \le \pi$
$a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$-a^n u[-n-1], \lvert a\rvert > 1$	$\dfrac{1}{1-ae^{-j\Omega}}$
$(n+1)a^n u[n], \lvert a\rvert < 1$	$\dfrac{1}{(1-ae^{-j\Omega})^2}$
$a^{\lvert n\rvert}, \lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\Omega+a^2}$
$x[n]=\begin{cases}1 & \lvert n\rvert \le N_1 \\ 0 & \lvert n\rvert > N_1\end{cases}$	$\dfrac{\sin\left[\Omega\left(N_1+\frac{1}{2}\right)\right]}{\sin(\Omega/2)}$
$\dfrac{\sin Wn}{\pi n}, 0 < W < \pi$	$X(\Omega)=\begin{cases}1 & 0 \le \lvert\Omega\rvert \le W \\ 0 & W < \lvert\Omega\rvert \le \pi\end{cases}$
$\sum_{k=-\infty}^{\infty}\delta[n-kN_0]$	$\Omega_0\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_0), \Omega_0=\dfrac{2\pi}{N_0}$

Example 4.5

From table 5.2, identified the Fourier transform of the following signals:

a) $x(t) = 1$

b) $x(t) = e^{j\omega 0t}$

c) $x(t) = e^{-j\omega 0t}$

d) $x(t) = \cos \omega_0 t$

e) $x(t) = \sin \omega_0 t$

Solution

TABLE 5.2 Common Fourier Transforms Pairs

$x(t)$	$X(\omega)$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$\sin \omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
$e^{-at}u(t), a > 0$	$\frac{1}{j\omega + a}$
$te^{-at}u(t), a > 0$	$\frac{1}{(j\omega + a)^2}$
$e^{-a\|t\|}, a > 0$	$\frac{2a}{a^2 + \omega^2}$
$\frac{1}{a^2 + t^2}$	$e^{-a\|\omega\|}$
$e^{-at^2}, a > 0$	$\sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$
$p_a(t) = \begin{cases} 1 & \|t\| < a \\ 0 & \|t\| > a \end{cases}$	$2a \frac{\sin \omega a}{\omega a}$
$\frac{\sin at}{\pi t}$	$p_a(\omega) = \begin{cases} 1 & \|\omega\| < a \\ 0 & \|\omega\| > a \end{cases}$
$\operatorname{sgn} t$	$\frac{2}{j\omega}$
$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0), \omega_0 = \frac{2\pi}{T}$

FREQUENCY RESPONSE

$$Y(\omega) = X(\omega)H(\omega)$$

The frequency response of the system is denoted by H().

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

The discrete Fourier series representation of a periodic sequence *x[n]* with fundamental period N_o is given

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{jk\Omega_0 n} \qquad \Omega_0 = \frac{2\pi}{N_0}$$

A periodic signal x(t) with fundamental period To has a complex exponential Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Example 4.6

A causal discrete time LTI system is describe by

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = x[n]$$

The system's input and output are x[n] and y[n], respectively.
a) Determine the frequency response H(Ω) of the system
b) Find the impulse response h[n] of the system

Solution

(a) Taking the Fourier transform of Eq. (6.143), we obtain

$$Y(\Omega) - \frac{3}{4}e^{-j\Omega}Y(\Omega) + \frac{1}{8}e^{-j2\Omega}Y(\Omega) = X(\Omega)$$

or

$$\left(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}\right)Y(\Omega) = X(\Omega)$$

Thus,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}} = \frac{1}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

(b) Using partial-fraction expansions, we have

$$H(\Omega) = \frac{1}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)} = \frac{2}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

Taking the inverse Fourier transform of $H(\Omega)$, we obtain

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$$

- Fourier and z-transforms are related through

$$z = e^{j\omega T}$$

$$X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

	Sequence	z - transform
1	$\delta[n]$	1
2	u[n]	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$

Example 4.7

Consider a discrete time LTI system is describe by

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] + \tfrac{1}{2}x[n-1]$$

a) Determine the frequency response H(Ω) of the system
b) Find the impulse response h[n] of the system

Solution

a)

$$Y(\Omega) - \frac{1}{2}e^{-j\Omega}Y(\Omega) = X(\Omega) + \frac{1}{2}e^{-j\Omega}X(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

b)

$$H(\Omega) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} + \frac{1}{2}\,\frac{e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

The inverse Fourier transform of H(Ω),

$$h[n] = \left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1] = \begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^{n-1} & n \geq 1 \end{cases}$$

Example 4.8

Show that the frequency response of the discrete time system

$$y[n-2] + 5y[n-1] + 6y[n] = 8x[n-1] + 18x[n]$$

is

$$H(e^{-j\Omega}) = \frac{8e^{-j\Omega} + 18}{(e^{-j\Omega})^2 + 5e^{-j\Omega} + 6}$$

Solution

$$y[n-2] + 5y[n-1] + 6y[n] = 8x[n-1] + 18x[n]$$

Convert to Laplace Transform

$$s^2Y(s) + 5sY(s) + 6Y(s) = 8sX(s) + 18X(s)$$

$$Y(s)[s^2 + 5s + 6] = X(s)\,[8s + 18]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{8s + 18}{s^2 + 5s + 6}$$

Substitute the s = $e^{-j\Omega}$

$$H(e^{-j\Omega}) = \frac{8e^{-j\Omega} + 18}{(e^{-j\Omega})^2 + 5e^{-j\Omega} + 6}$$

Sometimes the signs and signals of the non-language speaking world are not very clear. Then we must walk in trust, move forward step by small step, until we are sure of the proper path.

~Elaine Seiler~

TUTORIAL

Q1. Consider the difference equation for a three-point moving-average discrete-time filter.

$$y[n] = -\{x[n] + x[n-1] + x[n-2]\}$$

(a) Find and sketch the impulse response h[n] of the filter.
(b) Find the frequency response $H(\Omega)$ of the filter.
(c) Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$ of the filter.

Q2 . Consider a causal discrete-time FIR filter described by the impulse response

$$h[n] = \{2, 2, -2, -2\}$$

(a) Sketch the impulse response $h[n]$ of the filter.
(b) Find the frequency response $H(\Omega)$ of the filter.
(c) Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$ of the filter.

TUTORIAL

Q3. Determine the discrete Fourier series representation for each of the following sequences:

(a) $x[n] = \cos\frac{\pi}{4}n$

(b) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$

(c) $x[n] = \cos^2\left(\frac{\pi}{8}n\right)$

Q4. Let x[n] be a real periodic sequence with fundamental period N_0 and Fourier coefficients $c_k = a_k + jb_k$, where a_k and b_k are both real.

(a) Show that $a_{-k} = a_k$ and $b_{-k} = -b_k$.

(b) Show that $c_{N_0/2}$ is real if N_0 is even.

REFERENCES

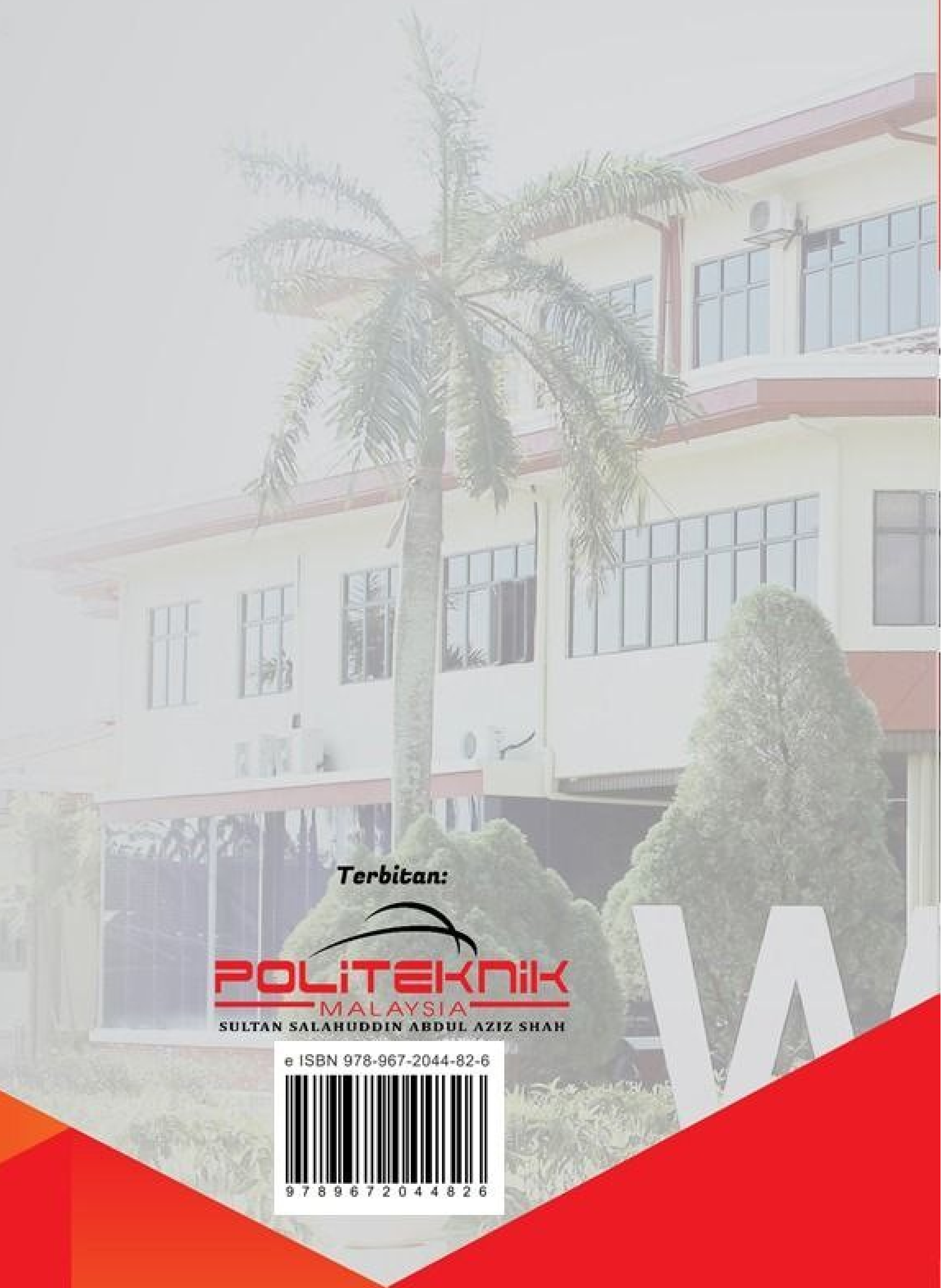
1. Hwei P.Hsu (2011). SIGNAL AND SYSTEM 2nd EDISION. New Jersey: McGRAW-HILL

2. Hwei P. Hsu, Ph.D (2014). Schaum's Outline of Signals and Systems. 3rd Edition. The McGraw-Hill Companies, Inc.

3. Michael Weeks (2010). Digital Signal Processing Using MATLAB & Wavelets. 2nd Edition. Jones and Bartlett Publishers, Inc.

4. Alan V. Oppenheim, Alan S. Willsky and S. Hamid Nawab (2015). Signals and Systems. Pearson Education Limited

5. Monson Hayes (2011). Schaums Outline of Digital Signal Processing. 2nd Edition. Mcgraw-Hill



Terbitan:

POLITEKNIK MALAYSIA
SULTAN SALAHUDDIN ABDUL AZIZ SHAH

e ISBN 978-967-2044-82-6

9 789672 044826