ORDINARY DIFFERENTIAL EQUATIONS

VOLUME 1
1ST EDITION

WRITTEN BY
RABIATUL ADAWIYAH BINTI ROSLI
ZURAIDAH BINTI OMAR
NUR RAIHAN BINTI ABDUL SALIM

JABATAN MATEMATIK, SAINS & KOMPUTER
POLITEKNIK SULTAN SALAHUDDIN ABDUL AZIZ SHAH

Politeknik Sultan Salahuddin Abdul Aziz Shah
Shah Alam, Selangor

Ordinary Differential Equations
Engineering Mathematics 3

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RABIATUL ADAWIYAH BINTI ROSLI
ZURAIDAH BINTI OMAR
NUR RAIHAN BINTI ABDUL SALIM

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Synopsis

This module contains notes, examples and exercises of material given as a course on Ordinary Differential Equations (ODEs) which is developed and revise based on Topic 3 DBM30033, Engineering Mathematics 3 and DBM30043, Electrical Engineering Mathematics Polytechnic Course Syllabus. All the methods given in the book are explained with the help of solved examples. The book begins with a introduction of differential equations, defines basic terms and outlines the general solution of a differential equation. We really hope that this module is beneficial to assist students to understand the subject more.

Preface

Thanks to Allah S.W.T for granting us strength and time to accomplish this Ordinary Differential Equations book. We also would like to thanks to those who were involved directly or indirectly in making this module success.

Congratulation to all the writers from the department of Mathematics, Science and Computer, Polytechnic Sultan Salahuddin Abdul Aziz Shah who were involve in writing this module:

Rabiatul Adawiyah binti Rosli
Zuraidah binti Omar
Nur Raihan binti Abdul Salim

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Key to symbols in this book

This symbol means that you want to discuss a point with your teacher. If you are working on your own there are answers in the back of the book. It is important, however, that you have a go at answering the questions before looking up theanswers if you are to understand the mathematics fully.

This is a warning sign. It is used where a common mistake, misunderstanding ortricky point is being described.

1 Ordinary Differential Equation

Sherlock Holmes: 'Now the skillful workman is very careful indeed … He will have nothing but the tools which may help him in doing his work, but of these he has a large assortment, and all in the most perfect order.'

A. Conan Doyle

By the end of this chapter you should be able to:

- Familiarize with and classify the Differential Equation
- Explain the form of the Differential Equation
- Solve the First Order Differential Equation by using method of:
  - (a) Direct integration
  - (b) Variable Separable
  - (c) Substitution y=vx (Homogenous Equations)
  - (d) Integrating Factor (for Linear Equations)
- Solve the Second Order Differential Equation if the auxiliary equations have:
  - (a) Real and Different Roots where $b^2 > 4ac$
  - (b) Real and Equal Roots where $b^2 = 4ac$
  - (c) Imaginary Roots where $b^2 < 4ac$
- Solve Particular Solution of First and Second Order Differential Equation

1.0 INTRODUCTION

An equation that contains a derivative (or derivatives) of an unknown function is called a differential equation. It is said to be an **ordinary differential equation** if all derivatives are with respect to a **single** independent **variable**, such as

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \cdots, \quad \frac{d^ny}{dx^n}$$

The differential equation is said to be partial if there are derivatives with respect to **two or more independent variables**, such as

$$\frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial x \partial y}, \quad \ldots.$$

1.1 FAMILIARIZE WITH AND CLASSIFY DIFFERENTIAL EQUATIONS

Basic Definition of the Differential Equation

A Differential Equation is any equation which contains derivatives, either ordinary derivatives or partial derivatives.

Order

The order of a differential equation is **the highest order of the derivative** present in the differential equation.

Example of first order → $8\boxed{\frac{dy}{dx}} + 3y = 6x$

Example of second order → $\boxed{\frac{d^2y}{dx^2}} + 6\frac{dy}{dx} + 9y = -2e^{3x}$

Degree

The degree of a differential equation is **the highest power of the highest derivatives** which occurs in the Differential Equation

Example of **first order** $\frac{dy}{dx}$ and **first degree** $\left(\frac{dy}{dx}\right)^1$ → $8\frac{dy}{dx}+3y=6x$

Example of **second order** $\frac{d^2y}{dx^2}$ and **third degree** $\left(\frac{d^2y}{dx^2}\right)^3$ → $\left(\frac{d^2y}{dx^2}\right)^3+6\frac{dy}{dx}+9y=-2e^{3x}$

EXAMPLE 1

State the dependent variable, independent variable, order, and degree of the Differential Equation below:

(i) $\left(\frac{d^3y}{dx^3}\right)^2+\sin y\left(\frac{dy}{dx}\right)=e^x$

(ii) $\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=x$

(iii) $\frac{dt}{ds}=(ts)^3$

.

SOLUTION

(i) $\left(\frac{d^3y}{dx^3}\right)^2+\sin y\left(\frac{dy}{dx}\right)=e^x$

The dependent variable is y and independent variable is x.

This DE has **order 3** (the highest derivative appearing is the **third** derivative) and **degree 2** (the **power** of the highest derivative is 2.)

(ii) $\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = x$

The dependent variable is y and independent variable is x.

This DE has **order 2** (the highest derivative appearing is the **second** derivative) and **degree 1** (the **power** of the highest derivative is 1.)

(iii) $\frac{dt}{ds} = (ts)^3$

The dependent variable is t and independent variable is s.

This DE has **order 1** (the highest derivative appearing is the **first** derivative) **degree 1** (the **power** of the highest derivative is 1.)

1.2 FORM OF DIFFERENTIAL EQUATION

Differential Equation can occur when arbitrary constant are eliminated from the given function. They follow the rule below:

- **1st order** Differential Equation is derived from a function having **1 arbitrary constant**.
- **2nd order** Differential Equation is derived from a function having **2 arbitrary constants**.

Therefore, an **n-th order** Differential Equation is derived from a function having **'n' arbitrary constants**.

EXAMPLE 2

Form the differential equation, where A, B, C and D are arbitrary constants:

(i) $y = Ae^{3x}$

(ii) $y = Ax^2 + 3Bx$

(iii) $y = C\cos x + D\sin x$

SOLUTION

(i) $y = Ae^{3x}$ .... (1)

- Differentiate the equation, $\dfrac{dy}{dx} = 3Ae^{3x}$ .... (2)
- Rearrange (2) so that, $Ae^{3x} = \dfrac{1}{3}\dfrac{dy}{dx}$ .... (3)
- Then substitute (3) into (1), $y = \dfrac{1}{3}\dfrac{dy}{dx}$

1st order, 1st degree

**Note: Function has 1 arbitrary constant, differentiate 1 time to eliminate the arbitrary constant*

(ii) $y = Ax^2 + 3Bx$ .... (1)

- Differentiate the equation (1),

$$\frac{dy}{dx} = 2Ax + 3B \quad \text{.................... (2)}$$

- And again differentiate (2),

$$\frac{d^2y}{dx^2} = 2A \quad \text{.................... (3)}$$

- Rearrange (2) so that,

$$3B = \frac{dy}{dx} - 2Ax \quad \text{.................... (4)}$$

- Rearrange (3) so that,

$$A = \frac{1}{2}\frac{d^2y}{dx^2} \quad \text{.................... (5)}$$

- Then substitute (4) and (5) into (1),

$$y = \left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x^2 + \left[\frac{dy}{dx} - 2\left(\frac{1}{2}\frac{d^2y}{dx^2}\right)x\right]x$$

$$= \frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2}$$

$$= -\frac{x^2}{2}\frac{d^2y}{dx^2} + x\frac{dy}{dx}$$

2^{nd} order, 1^{st} degree

**Note: Function has 2 arbitrary constant, differentiate 2 times to eliminate the arbitrary constant*

(iii) $y = C\cos x + D\sin x$ .... (1)

- Differentiate the equation (1),

$$\frac{dy}{dx} = -C\sin x + D\cos x \quad \text{.................... (2)}$$

- And again differentiate (2),

$$\frac{d^2y}{dx^2} = -C\cos x - D\sin x \quad \text{.................... (3)}$$

- Rearrange (3) so that,

$$\frac{d^2y}{dx^2} = -(C\cos x + D\sin x) \quad \text{.................... (4)}$$

- Then substitute (1) into (4),

$$\frac{d^2y}{dx^2} = -(y)$$

$$y = -\frac{d^2y}{dx^2}$$

2nd order, 1st degree

**Note: Function has 2 arbitrary constant, differentiate 2 times to eliminate the arbitrary constant*

TEST YOURSELF

Form a differential equation for each of the following functions:

a. $y = Ax^3 + x^4$

b. $y = Ax^4 + 7x - 9$

c. $y = Ax^2 - Bx + x$

d. $y = A\cos(3x + B)$

e. $y = Ax + \dfrac{B}{x}$

f. $y = Ae^{3x} - 6Be^{3x}$

CHECK YOUR ANSWER

a. $x\dfrac{dy}{dx} = 3y + x^4$

b. $x\dfrac{dy}{dx} = 4y - 21x + 36$

c. $y = -\dfrac{x^2}{2}\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$

d. $\dfrac{d^2y}{dx^2} = -9y$

e. $y = x\dfrac{dy}{dx} + x^2\dfrac{d^2y}{dx^2}$

f. $\dfrac{d^2y}{dx^2} = 9y$

1.3 SOLUTION OF FIRST ORDER DIFFERENTIAL EQUATION

A solution of an ordinary differential equation is a function that satisfies differential equation, which makes the equation true (left-hand side equal to right-hand side) by manipulate the equation so as to **eliminate all the derivatives** and **leave a relationship between** y **and** x.

There are four methods to solve the differential equation

Method 1	Method 2	Method 3	Method 4
Direct Integration	Variable Separable	Substitution of y=vx	Integration Factor

1.3.1 DIRECT INTEGRATION

If the equation can be arranged in the form of $dy = f(x)\,dx$, then the equation can be solved by **simple integration**, where

$$\int dy = \int f(x)dx$$

EXAMPLE 3

Solve the following differential equation:

(i) $\frac{dy}{dx} = 3x^2 - 6x + 5$

(ii) $2y' = sin5x$

(iii) $\frac{dy}{dx} - 5x = 0$

(iv) $\frac{dy}{dx} = x^2 - e^{\frac{x}{4}}$

(v) $x\frac{dy}{dx} = x^2 + 2x - 3$

(vi) $y'e^{-x} + e^{2x} = 0$

SOLUTION

(i) $\frac{dy}{dx} = 3x^2 - 6x + 5$

- Rearrange equation, $dy = 3x^2 - 6x + 5 \cdot dx$
- Integrate both sides, $\int dy = \int (3x^2 - 6x + 5) \cdot dx$

$$\therefore \; y = \frac{3x^{2+1}}{3} - \frac{6x^{1+1}}{2} + 5x + c = x^3 - 3x^2 + 5$$

(ii) $2y' = sin5x$

- Rearrange equation,

$$\frac{dy}{dx} = sin5x$$

$$dy = \frac{1}{2} sin5x.dx$$

- Integrate both sides,

$$\int dy = \int \frac{1}{2} sin5x.dx$$

$$= -\frac{1}{2}\frac{cos5x}{5} + c = -\frac{cos5x}{10} + c$$

(iii) $\dfrac{dy}{dx} - 5x = 0$

- Rearrange equation,

$$\frac{dy}{dx} = 5x$$

$$dy = 5x \cdot dx$$

- Integrate both sides,

$$\int dy = \int 5x \cdot dx$$

$$\therefore \quad y = \frac{5x^{1+1}}{2} + c = \frac{5x^2}{2} + c$$

(iv) $\dfrac{dy}{dx} = x^2 - e^{\frac{x}{4}}$

Rearrange equation

$$dy = x^2 - e^{\frac{x}{4}} \cdot dx$$

Integrate both sides,

$$\int dy = \int \left(x^2 - e^{\frac{x}{4}}\right) \cdot dx$$

$$\therefore \quad y \quad \frac{x^{2+1}}{3} - \frac{e^{\frac{x}{4}}}{\frac{1}{4}} + c = \frac{x^3}{3} - 4e^{\frac{x}{4}} + c$$

(v) $x\dfrac{dy}{dx} = x^2 + 2x - 3$

- Rearrange and simplify equation,

$$dy = \frac{x^2+2x-3}{x} \cdot dx \quad = x + 2 - \frac{3}{x} \cdot dx$$

- Integrate both sides,

$$\int dy = \int \left(x + 2 - \frac{3}{x}\right) \cdot dx$$

$$\therefore \quad y = \frac{x^{1+1}}{2} + \frac{2x^{0+1}}{1} - 3\,ln|x| + c$$

$$= \frac{x^2}{2} + 2x - 3\,ln|x| + c$$

(vi) $y'e^{-x} + e^{2x} = 0$

- Rearrange and simplify equation,

$$\frac{dy}{dx} e^{-x} + e^{2x} = 0$$

<u>Law of Exponent</u>: $e^{2x-(-x)} = e^{3x}$

$$dy = \frac{-e^{2x}}{e^{-x}} \cdot dx$$

$$= -e^{3x} \cdot dx$$

- Integrate both sides,

$$\int dy = \int -e^{3x} \cdot dx$$

$$\therefore \quad y = \frac{-e^{3x}}{3} + c$$

Please click link below to refer example 3(i) video solution
https://www.youtube.com/watch?v=fWTr8IYJuaQ

1.3.2 VARIABLE SEPARABLE

If the given equation is in form $\frac{dy}{dx}=f(x)\cdot g(y)$ or $\frac{dy}{dx}=\frac{f(x)}{g(y)}$, and can be expressed and reduced as shown below,

$$\frac{dy}{dx}=f(x)\cdot g(y) \quad\Rightarrow\quad \frac{1}{g(y)}\cdot dy=f(x)\cdot dx$$

$$\frac{dy}{dx}=\frac{f(x)}{g(y)} \quad\Rightarrow\quad g(y)\cdot dy=f(x)\cdot dx$$

where variable x appears on one side (right-side) and variable y appears on the other side (left-side), such a differential equation is called a separable differential equation, where

$$\int\frac{1}{g(y)}\cdot dy=\int f(x)\cdot dx$$

or

$$\int g(y)\cdot dy=\int f(x)\cdot dx$$

? How to separate the variables correctly? You need to know the proper way to transit the variable expression correctly.

EXAMPLE 4

Solve the following differential equation $\frac{dy}{dx} = \frac{2x}{y+1}$.

SOLUTION

$g(y) = y+1$ $f(x) = 2x$

- Separate the expression of x and y,

$$y+1 \cdot dy = 2x \cdot dx$$

- Integrate both sides,

$$\int (y+1) \cdot dy = \int 2x \cdot dx$$

$$\therefore \frac{y^2}{2} + y = \frac{2x^2}{2} + c$$

$$\frac{y^2}{2} + y = x^2 + c$$

EXAMPLE 5

Solve the following differential equation $\frac{dy}{dx} = (1+x)(1+y)$

SOLUTION

- Separate the expression of x and y,

$$\frac{1}{1+y} \cdot dy = 1 + x \cdot dx$$

- Integrate both sides,

$$\int \frac{1}{1+y} \cdot dy = \int (1+x) \cdot dx$$

$$\therefore \ln|1+y| = x + \frac{x^2}{2} + c$$

EXAMPLE 6

Solve the following differential equation $\frac{dy}{dx} = xy - y$.

SOLUTION

- Separate the expression of x and y,

$$\frac{dy}{dx} = y(x-1)$$

$$\frac{1}{y} \cdot dy = (x-1) \cdot dx$$

- Integrate both sides,

$$\int \frac{1}{y} \cdot dy = \int (x-1) \cdot dx$$

$$\therefore \ln|y| = \frac{x^2}{2} - x + c$$

EXAMPLE 7

Solve the following differential equation $\frac{dy}{dx} = 2x^3 \cdot e^{-2y}$.

SOLUTION

- Separate the expression of x and y,

$$e^{2y} \cdot dy = 2x^3 \cdot dx$$

- Integrate both sides,

$$\int e^{2y} \cdot dy = \int 2x^3 \cdot dx$$

$$\therefore \frac{e^{2y}}{2} = \frac{2x^4}{4} + c$$

$$\frac{e^{2y}}{2} = \frac{x^4}{2} + c$$

EXAMPLE 8 Solve the following differential equation $\frac{dy}{dx} = \frac{y^2 - xy^2}{x^2 y + x^2}$.

SOLUTION

- Separate the expression of x and y,

$$\frac{dy}{dx} = \frac{y^2(1-x)}{x^2(y+1)}$$

$$\frac{y+1}{y^2} \cdot dy = \frac{1-x}{x^2} \cdot dx$$

$$\frac{y}{y^2} + \frac{1}{y^2} \cdot dy = \frac{1}{x^2} - \frac{x}{x^2} \cdot dx$$

$$\frac{1}{y} + y^{-2} \cdot dy = x^{-2} - \frac{1}{x} \cdot dx$$

- Integrate both sides,

$$\int\left(\frac{1}{y} + y^{-2}\right) \cdot dy = \int\left(x^{-2} - \frac{1}{x}\right) \cdot dx$$

$$\therefore \ln|y| + \frac{y^{-1}}{-1} = \frac{x^{-1}}{-1} - \ln|x| + c$$

$$\ln|y| - \frac{1}{y} = \frac{-1}{x} - \ln|x| + c$$

*Note that, $\int \frac{1}{y} \cdot dy = \ln|y|$ but if $\int \frac{1}{y^2} \cdot dy \neq \ln|y^2|$

Therefore $\int \frac{1}{y^2} \cdot dy = \int y^{-2} \cdot dy = \frac{y^{-2+1}}{-2+1} = \frac{y^{-1}}{-1} = \frac{-1}{y}$

Please click link below to refer example 4 &8 video solution
https://www.youtube.com/watch?v=8P8i2A6GZ6Y

EXAMPLE 9

Solve the following differential equation $\frac{dy}{dx} = e^{2x-3y}$.

SOLUTION

- Separate the expression of x and y,

$$\frac{dy}{dx} = \frac{e^{2x}}{e^{3y}}$$

$$e^{3y} \cdot dy = e^{2x} \cdot dx$$

- Integrate both sides,

$$\int e^{3y} \cdot dy = \int e^{2x} \cdot dx$$

$$\therefore \frac{e^{3y}}{3} = \frac{e^{2x}}{2} + c$$

1.3.3 SUBSTITUTION Y=VX

For any ordinary differential equation of $\frac{dy}{dx} = f(x, y)$, if $f(x, y) = f(\lambda x, \lambda y)$, where λ is the real number, the equation is called a homogeneous differential equation. This is determined by the fact that the total degree in x and y for each of the terms involved is the same.

Example: $\frac{dy}{dx} = \frac{x+3y}{2x}$

Condition of the equation:

(i) Total degree is 1 for x term and y term → Homogeneous DE

(ii) The variables x and y cannot be separated → Doesn't fit to solve using Variable Separable Method

Therefore, the key to solve every homogeneous equation is to substitute,

$y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

This converts the equation into a form which can be solved by separating the variables.

How to identify that problem could be solved by Variable Separable technique?

EXAMPLE 10

Solve the following differential equation $\frac{dy}{dx} = \frac{x+3y}{2x}$.

SOLUTION

- Substitute $y \to vx$

and $\frac{dy}{dx} \to v + x\frac{dv}{dx}$

then simplify,

$$v + x\frac{dv}{dx} = \frac{x+3(vx)}{2x}$$

$$x\frac{dv}{dx} = \frac{x+3vx}{2x} - v$$

$$= \frac{x+3vx-2vx}{2x}$$

$$= \frac{x+vx}{2x}$$

$$= \frac{x(1+v)}{x(2)} = \frac{1+v}{2}$$

- Separate the expression of x (right-side) and v (left-side),

$$\frac{1}{1+v}\cdot dv = \frac{1}{2}\cdot\frac{1}{x}\cdot dx$$

- Integrate both sides,

$$\int \frac{1}{1+v}\cdot dv = \frac{1}{2}\int \frac{1}{x}\cdot dx$$

$$\ln|1+v| = \frac{1}{2}\ln|x| + c$$

- Since $y = vx$

therefore substitute $v \to \frac{y}{x}$,

$$\ln\left|1+\frac{y}{x}\right| = \frac{1}{2}\ln|x| + c$$

EXAMPLE 11

Solve the following differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$.

SOLUTION

- Substitute $y \to vx$

and $\frac{dy}{dx} \to v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x(vx)}$$

then simplify,

$$\begin{aligned} x\frac{dv}{dx} &= \frac{x^2 + v^2x^2}{x^2v} - v \\ &= \frac{x^2 + v^2x^2 - v^2x^2}{x^2v} \\ &= \frac{x^2(1)}{x^2(v)} \\ &= \frac{1}{v} \end{aligned}$$

- Separate the expression of x (right-side) and v (left-side),

$$v \cdot dv = \frac{1}{x} \cdot dx$$

- Integrate both sides,

$$\begin{aligned} \int v \cdot dv &= \int \frac{1}{x} \cdot dx \\ \frac{v^2}{2} &= \ln|x| + c \end{aligned}$$

- Since $y = vx$

therefore substitute $v \to \frac{y}{x}$,

$$\begin{aligned} \frac{1}{2}\left(\frac{y}{x}\right)^2 &= \ln|x| + c \\ \frac{y^2}{2x^2} &= \ln|x| + c \end{aligned}$$

EXAMPLE 12 Solve the following differential equation $\left(x^2+xy\right)\frac{dy}{dx} = xy - y^2$.

SOLUTION

- Rearrange equation,

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2+xy}$$

- Substitute $y \to vx$ and $\frac{dy}{dx} \to v + x\frac{dv}{dx}$,

$$v + x\frac{dv}{dx} = \frac{x(vx)-(vx)^2}{x^2 + x(vx)}$$

then simplify,

$$v + x\frac{dv}{dx} = \frac{x^2v - v^2x^2}{x^2 + x^2v}$$

$$= \frac{x^2\left(v - v^2\right)}{x^2(1+v)}$$

$$x\frac{dv}{dx} = \frac{v - v^2}{1+v} - v$$

$$= \frac{v - v^2 - v - v^2}{1+v}$$

$$= \frac{-2v^2}{1+v}$$

- Separate the expression of x (right-side) and v (left-side),

$$\frac{1+v}{v^2}\cdot dv = -2\cdot\frac{1}{x}\cdot dx$$

- Integrate both sides,

$$\int\left(\frac{1}{v^2} + \frac{v}{v^2}\right)\cdot dv = -2\int\frac{1}{x}\cdot dx$$

$$\int\left(v^{-2} + \frac{1}{v}\right)\cdot dv = -2\int\frac{1}{x}\cdot dx$$

$$\frac{-1}{v} + \ln|v| = -2\ln|x| + c$$

- Since $y = vx$

therefore substitute $v \to \dfrac{y}{x}$,

$$\frac{-1}{\left(\frac{y}{x}\right)} + \ln\left|\frac{y}{x}\right| = -2\ln|x| + c$$

$$\frac{-x}{y} + \ln\left|\frac{y}{x}\right| = -2\ln|x| + c$$

1.3.4 INTEGRATING FACTOR

The differential equation of the form $\frac{dy}{dx} + Py = Q$ is called linear equation of the first order, where P and Q are constants or functions of x. Any such equation can be solved by multiplying both sides by an integrating factors (IF). These are steps in solving first order differential equation by using Integrating Factor.

STEP 1

Write the given equation in the form of :

$$\frac{dy}{dx} + Py = Q$$

STEP 2

Find the Integrating Factor (IF) by using :

$$I.F = e^{\int P\,dx}$$

STEP 3

Solve equation given by using formula :

$$y \cdot IF = \int Q \cdot IF\,dx$$

EXAMPLE 13

Solve the following differential equation $\frac{dy}{dx}+5y=e^{2x}$

SOLUTION

- Form $\frac{dy}{dx}+Py=Q$,
- Then identify P and Q,

$$\frac{dy}{dx}+5y = e^{2x}$$

$$\Rightarrow \quad P = 5, \; Q = e^{2x}$$

- Find IF by substitute P,

$$I.F = e^{\int P(x)dx} = e^{\int 5\cdot dx} = e^{5x}$$

- Solve using formula,

by substitute Q and IF,

$$y\cdot IF = \int Q\cdot .IF\cdot dx$$

$$y\cdot e^{5x} = \int e^{2x}\cdot .e^{5x}\cdot dx$$

$$= \int e^{7x}dx$$

$$y\cdot e^{5x} = \frac{e^{7x}}{7}+c$$

$$\therefore y = \frac{e^{7x}}{7e^{5x}}+\frac{c}{e^{5x}}$$

EXAMPLE 14

Solve the following differential equation $x\frac{dy}{dx}+y=x^3$

SOLUTION

- Form $\frac{dy}{dx}+Py=Q$,

$$\frac{x}{x}\frac{dy}{dx}+\frac{y}{x} = \frac{x^3}{x}$$

$$\frac{dy}{dx}+\frac{y}{x} = x^2$$

- Then identify P and Q,

$$\Rightarrow \quad P = \frac{1}{x}, \quad Q = x^2$$

- Find IF by substitute P,

$$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{x}\cdot dx} = e^{\ln|x|} = x$$

One of the Rule of exponent, $e^{\ln a} = a$

- Solve using formula

by substitute Q and IF,

$$y\, IF = \int Q \cdot .IF \cdot dx$$

$$y\cdot x = \int x^2 \cdot x\, dx$$

$$y x = \int x^3\, dx$$

$$y x = \frac{x^4}{4}+c$$

$$\therefore y = \frac{x^4}{4x}+\frac{c}{x}$$

EXAMPLE 15 Solve the following differential equation $(x-2)\frac{dy}{dx} - y = (x-2)^3$

SOLUTION

- Form $\frac{dy}{dx} + Py = Q$,

$$\frac{\cancel{(x-2)}}{\cancel{(x-2)}}\frac{dy}{dx} - \frac{y}{(x-2)} = \frac{(x-2)^3}{\cancel{(x-2)}}$$

$$\frac{dy}{dx} - \frac{y}{(x-2)} = (x-2)^2$$

- Then identify P and Q,

$$\Rightarrow \quad P = \frac{-1}{x-2}, \quad Q = (x-2)^2$$

- Find IF by substitute P,

$$I.F = e^{\int P(x)dx} = e^{-\int \frac{1}{x-2} \cdot dx} = e^{-\ln|x-2|}$$

$$= e^{\ln|x-2|^{-1}}$$

$$= (x-2)^{-1}$$

$$= \frac{1}{x-2}$$

- Solve using formula,

by substitute Q and IF,

$$y \cdot IF = \int Q \cdot .IF \cdot dx$$

$$y \cdot \frac{1}{x-2} = \int (x-2)^{\cancel{2}} \cdot \frac{1}{\cancel{x-2}} \cdot dx$$

$$y \cdot \frac{1}{x-2} = \int (x-2) dx$$

$$\frac{y}{x-2} = \frac{x^2}{2} - 2x + c$$

EXAMPLE 16

Solve the following differential equation $\frac{dy}{dx} - y = x$.

SOLUTION

- Form $\frac{dy}{dx} + Py = Q$
- Then identify P and Q,

$$\frac{dy}{dx} - y = x$$

$$\Rightarrow \quad P = -1, \; Q = x$$

- Find IF by substitute P,

$$I.F = e^{\int P(x)dx}$$

$$= e^{\int -1 \cdot dx} = e^{-x}$$

- Solve using formula,
- by substitute Q and IF,

$$y \cdot IF = \int Q \cdot .IF \cdot dx$$

$$y \cdot e^{-x} = \int x \cdot e^{-x} \cdot dx$$

Integration of products
(Between polynomial and exponent)

By using Integration By Parts Method

(for LHS equation),

$$y \cdot e^{-x} = uv - \int v du$$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$$

$$dv = e^{-x} dx \quad \Rightarrow \quad v = \int e^{-x} dx = \frac{e^{-x}}{-1} = -e^{-x}$$

Therefore,

$$y \cdot e^{-x} = (x)\left(-e^{-x}\right) - \int \left(-e^{-x}\right)(dx)$$

$$= -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$= -xe^{-x} - e^{-x} + c$$

$$\therefore\ y = \frac{-xe^{-x}}{e^{-x}} - \frac{e^{-x}}{e^{-x}} + \frac{c}{e^{-x}}$$

$$y = -x - 1 + \frac{c}{e^{-x}}$$

TEST YOURSELF Solve the following ordinary differential equation.

What is the suitable method for the following Ordinary Differential Problems? How to identify the suitable method?

i. $\frac{dy}{dx} = 8x^3y^2$

ii. $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$

c. $\frac{dy}{dx} + 2xy = x$

d. $x\frac{dy}{dx} = \frac{4y}{y-3}$

e. $\frac{dy}{dx} + 2y = e^{2x}$

f. $\frac{dy}{dx} = 3x^2y^2$

g. $x^2(1-y)\frac{dy}{dx} = (1+x)y$

h. $\frac{dy}{dx} + y\tan x = \sin 2x$

i. $2xy\frac{dy}{dx} = y^2 - x^2$

j. $xy\frac{dy}{dx} = \frac{(x^2-1)}{(y-1)}$

k. $x^2(1-y)\frac{dy}{dx} = (1+x)y$

l. $x\frac{dy}{dx} + y = x^3$

CHECK YOUR ANSWER

a. $-\frac{1}{y} = 2x^4 + c$

b. $\frac{y^2}{2x^2} = \ln x + c$

c. $\frac{-ln(1-2y)}{2} = \frac{x^2}{2} + c$

d. $\frac{1}{4}y - \frac{3}{4}\ln y = \ln x + c$

e. $y = \frac{e^{2x}}{4} + C$

$-\frac{1}{y} = x^3 + c$

g. $(\ln y) - y = -\frac{1}{x} + \ln x + c$

h. $y = -2\cos^2 x + C$

i. $-\ln\left(\left(\frac{y}{x}\right)^2 + 1\right) = \ln x + c$

j. $2y^3 - 3y^2 = 3x^2 - 6\ln x + C$

k. $(\ln y) - y = -\frac{1}{x} + \ln x + c$

l. $y = \frac{x^3}{4} + C$

1.4 SECOND ORDER OF DIFFERENTIAL EQUATIONS

The **general form** of second order differential equation with constant is

$a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + c\,y = 0$ and the **Auxiliary Equation** is

$$a\,m^2 + b\,m + c = 0$$

where a, b and c are constants with $a > 0$ and $m^2 = \dfrac{d^2y}{dx^2}$, $m = \dfrac{dy}{dx}$, $y = 1$

1.4.1 SOLVE GENERAL SOLUTION OF 2ND ODE

Solve the Second Order Differential Equation have:

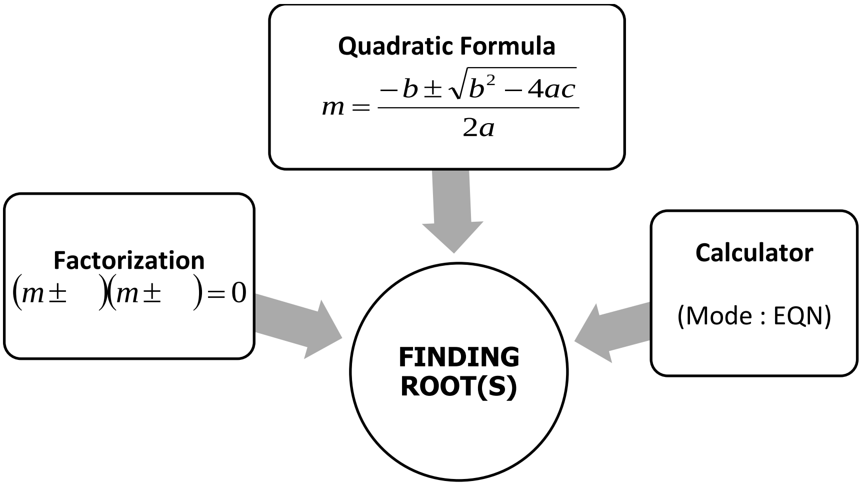
(a) Real and Different Roots where $b^2 > 4ac$

(b) Real and Equal Roots where $b^2 = 4ac$

(c) Imaginary Roots where $b^2 < 4ac$

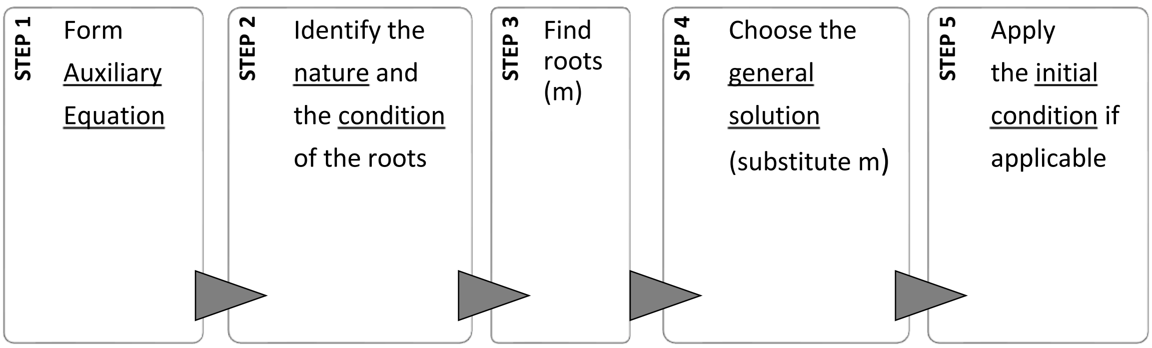
Nature of Roots	Condition	General Solution	Roots
Real and Different Roots	$b^2 - 4ac > 0$	$y = Ae^{m_1 x} + Be^{m_2 x}$	$m = m_1$ $m = m_2$
Real and Equal Roots	$b^2 - 4ac = 0$	$y = e^{mx}(A + Bx)$	$m = m_1 = m_2$
Complex Roots ?	$b^2 - 4ac < 0$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$	$m = \alpha \pm j\beta$

Finding Root(s)



What type of roots will use factorization or quadratic formula methods?

These are steps in solving second order differential equation



Please click link below to refer introduction second order of differential equation
https://youtu.be/Ahl0pI9aJuU

EXAMPLE 17

Determine the general solution for $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

SOLUTION

- Form Auxiliary Equation,

  $m^2+5m+6=0$

- Nature and condition,

  Real and Different Roots,

  $b^2-4ac = (5)^2-4(1)(6) = 1 > 0$

- Find the root(s),

  By using factorization,

  $(m+3)(m+2)=0$

  $\Rightarrow m_1=-3\,,\ m_2=-2$

- General Solution,

  (substitute m_1 and m_2)

  $y=Ae^{m_1x}+Be^{m_2x}$

  $\therefore\ y=Ae^{-3x}+Be^{-2x}$

  or,

  $y=Ae^{-2x}+Be^{-3x}$

  (for $m_1=-2$ and $m_2=-3$*)*

Please click link below to refer example 17 video solution
https://youtu.be/zfVTgvzzsQ0

EXAMPLE 18 Determine the general solution for $y'' + 6y' + 9y = 0$

SOLUTION

- Form Auxiliary Equation,

$$m^2 + 6m + 9 = 0$$

- Nature and condition,

Real and Equal Roots,

$$b^2 - 4ac = (6)^2 - 4(1)(9) = 0$$

- Find the root(s),

By using factorization,

$$(m+3)(m+3) = 0$$

$$\Rightarrow m_1 = m_2 = m = -3$$

- General Solution,

  (substitute *m*)

$$y = e^{mx}(A + Bx)$$

$$\therefore \quad y = e^{-3x}(A + Bx)$$

Please click link below to refer example 18 video solution
https://youtu.be/GXp4zvDrGHM

EXAMPLE 19

Determine the general solution for $2\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

SOLUTION

- Form Auxiliary Equation,

$$2m^2+m+1=0$$

Complex Roots,

- Nature and condition,

$$b^2-4ac \; = \; (1)^2-4(2)(1) \; = \; -7<0$$

- Find the root(s),

(cannot be factorized)

By using Quadratic Formula,

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{(1)^2-4(2)(1)}}{2(2)}$$

$$=\frac{-1\pm\sqrt{-7}}{4}$$

$$=\frac{-1\pm\sqrt{7}\sqrt{-1}}{4}$$

$$=\frac{-1}{4}\pm\frac{\sqrt{7}}{4}i$$

$$=-0.25\pm0.661i$$

$$\Rightarrow \; \alpha=-0.25\,, \; \beta=0.66$$

- General Solution,

(substitute α *and* β)

$$y=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$

$$\therefore y=e^{-0.25x}(A\cos 0.66x+B\sin 0.66x)$$

Please click link below to refer example 19 video solution
https://youtu.be/0IhawDpu6RI

TEST YOURSELF

Solve the following differential equations:

a. $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 41y = 0$

b. $y'' - 4y' - 4y = 0$

c. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

d. $2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 2y = 0$

e. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

f. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

g. $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25y = 0$

h. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

i. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$

j. $2y'' + 4y' + 3y = 0$

k. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 9y = 0$

l. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$

CHECK YOUR ANSWER

a. $y = e^{5x}(A\cos 4x + B\sin 4x)$

b. $y = Ae^{4.828x} + Be^{-0.828x}$

c. $y = Ae^{x} + Be^{-3x}$

d. $y = Ae^{-0.5x} + Be^{2x}$

e. $y = Ae^{-2x} + Be^{-3x}$

f. $y = Ae^{3x} + Be^{2x}$

g. $y = e^{-5x}(A + Bx)$

h. $y = e^{-2x}(A + Bx)$

i. $y = Ae^{4x} + Be^{-2x}$

j. $y = e^{-x}(A\cos 0.707x + B\sin 0.707x)$

k. $y = e^{-2x}(A\cos\sqrt{5}x + B\sin\sqrt{5}x)$

l. $y = e^{x}(A\cos 3x + B\sin 3x)$

1.4.2 SOLVE PARTICULAR SOLUTION OF 2ND ODE

EXAMPLE 20

Obtain the specific solution for the following differential equation:

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$, given that when $x = 0, y = 1$ and $\frac{dy}{dx} = 0$.

SOLUTION

- Form Auxiliary Equation,

  $m^2 + 3m - 4 = 0$

- Nature and condition,

  Real and Different Roots, $b^2 - 4ac = 3^2 - 4(1)(-4) = 25$

- Find the root(s),

  By using factorization, $(m + 4)(m - 1) = 0, m = -4,1$

- General Solution (substitute *m*),

  $y = Ae^{-4x} + Be^{x}$

  When $x = 0, y = 1$,

  $1 = A + B \ldots \ldots \ldots \ldots \ldots . . (1)$

  When $x = 0, \frac{dy}{dx} = 0$

  $\frac{dy}{dx} = -4Ae^{-4x} + Be^{x}$

  $0 = -4A + B$

  4A=B

  Substitute B=4A in (1),

  $1 = A + 4A$

  $1 = 5A$

  $A = \frac{1}{5}$

  $B = 4(\frac{1}{5})$

  $B = \frac{4}{5}$

  $y = \frac{1}{5}e^{-4x} + \frac{4}{5}e^{x}$

EXAMPLE 21

Solve the following equation in which s is the displacement of an object at time t,

$\frac{d^2s}{dt^2} - 4\frac{ds}{dt} + 4s = 0$. Given that $s = 1$, $\frac{ds}{dt} = 3$ when $t = 0$

SOLUTION

- Form Auxiliary Equation,

  $m^2 - 4m + 4 = 0$

- Nature and condition,

  Real and Equal Roots, $b^2 - 4ac = (-4)^2 - 4(1)(4) = 0$

- Find the root(s),

  By using factorization, $(m-2)(m-2) = 0 \quad \Rightarrow m = 2$

- General Solution (substitute *m*),

  $y = e^{mx}(A + Bx)$

  $\therefore \quad y = e^{2x}(A + Bx) \quad \Rightarrow \quad s = e^{2t}(A + Bt)$

- Condition $s(0) = 1$

  When $t = 0$,

  $$s = e^{2(0)}(A + B(0))$$
  $$1 = 1(A) \qquad \Rightarrow \therefore \; A = 1$$

- Condition $\frac{ds}{dt}(0) = 3$

  Differentiate s using product rule,

  $u = e^{2t} \Rightarrow u' = 2e^{2t} \quad and \quad v = A + Bt \Rightarrow v' = B$

  $$\frac{ds}{dt} = uv' + vu' = Be^{2t} + 2e^{2t}(A + Bt)$$

  When $t = 0$, and $A = 1$,

  $$\frac{ds}{dt} = Be^{2t} + 2e^{2t}(A + Bt)$$
  $$3 = Be^{2(0)} + 2e^{2(0)}(1 + B(0))$$
  $$3 = B + 2 \qquad \Rightarrow \therefore \; B = 1$$

- Particular Solution,

  Substitute $A = 1$ and $B = 1$ into s

  $s = e^{2t}(A + Bt) \quad \Rightarrow \quad s = e^{2t}(1 + t)$

TEST YOURSELF

Obtain the specific solution for the following differential equations

a. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, given that $x = 0, y = \frac{7}{2}$ and $\frac{dy}{dx} = 9$

b. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$, given that $y = 1, \frac{dy}{dx} = 0$ when $x = 0$

c. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, given that $y = \frac{7}{2}, \frac{dy}{dx} = 9$ when $x = 0$

CHECK YOUR ANSWER

a. $y = \frac{11}{2}e^{2x} - 2e^x$

b. $y = \frac{1}{5}e^{-4x} + \frac{4}{5}e^x$

c. $y = \frac{11}{2}e^{2x} - 2e^x$